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ON QUASI-P-ALMOST DISTRIBUTIVE LATTICES

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Abstract

In this paper, the concept of quasi pseudo-complementation on an Almost Distributive Lattice (ADL) as a generalization of pseudo-complementation on an ADL is introduced and its properties are studied. Necessary and sufficient conditions for a quasi pseudo-complemented ADL(q-p-ADL) to be a pseudo-complemented ADL(p-ADL) and Stone ADL are derived and the set $S(L) = \{a^* \mid a \in L\}$ is proved to be a Boolean algebra. Also, the notions of $*-$congruence and kernel ideals are introduced in a quasi-p-ADL and characterized kernel ideals. Finally, some equivalent conditions are given for every ideal of a quasi-p-ADL to be a kernel ideal.

Keywords: pseudo-complementation, quasi pseudo-complementation, Almost Distributive Lattice (ADL), p-ADL, quasi-p-ADL.

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1. Introduction

A pseudo-complemented lattice is a lattice L with 0 such that to each $a \in L$, the largest annihilating element of a exists in L. That is, there exists $a^* \in L$ such that, for all $x \in L, a \wedge x = 0$ if and only if $x \leq a^*$. Here a^* is called the pseudo-complement of a. For each element a of a pseudo-complemented lattice L, a^* is uniquely determined by a, so that $*$ can be regarded as a unary operation on L. Moreover, each pseudo-complemented lattice contains the unit element namely 0^*. It follows that every pseudo-complemented lattice L can be regarded as an algebra $(L, \vee, \wedge, *, 0, 1)$ of type $(2, 2, 1, 0, 0)$. The fact that the class of pseudo-complemented distributive lattices is equationally definable was first observed by Ribenboim in 1949. Also, in [5], it was proved that the class of pseudo-complemented distributive lattices is generated by its finite members and a complete description of the lattice of equational classes of pseudo-complemented distributive lattices is given. In [8], Sankappanavar introduced a new class of algebras, called semi-De Morgan algebras, as a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices and studied its properties. Also, he studied several important subvarieties of semi-De Morgan algebras, such as demi-p-lattices, weak Stone algebras and almost p-lattices. In [3], Frink studied about the pseudo-complemented semilattice L and proved that the set $L^* = \{a^* \mid a \in L\}$, where $*$ is a pseudo-complementation on L, becomes a Boolean algebra. In [1], Cornish considered the kernels of $*-$congruences on distributive pseudo-complemented lattices and studied its important properties. Later these concepts were extended to the case of semi lattices by Blyth in [2] and to the case of ADLs by Rao in [7].

The concept of pseudo-complementation in an ADL and the concept of Stone ADL was given by Swamy, Rao and Nanaji Rao [9, 10]. They have proved that there is a one-to-one correspondence between the pseudo-complementations on an ADL L with 0 and the set of all maximal elements of L. Also, they proved that if $*$ is a pseudo-complementation on an ADL L, then the set $L^* = \{a^* \mid a \in L\}$ is a Boolean algebra and the pseudo-complementation $*$ on L is equationally definable. In [6] Rao *et al.* studied the properties of minimal prime ideals in an ADL.

In this paper, we introduce the concept of quasi pseudo-complementation on an ADL as a generalization of pseudo-complementation on an ADL like the concept of almost p-lattice as a generalization of pseudo-complemented distributive lattice. Here we extend the concept of almost p-lattice to the case of almost distributive lattices and name it quasi-p-ADL. We give necessary and sufficient conditions for a quasi-p-ADL to be a p-ADL and we prove that if $*$ is a quasi pseudo-complementation on an ADL L then the set $S(L) = \{a^* \mid a \in L\}$ becomes a Boolean algebra. It is observed that there exists an induced surjective correspondence between the set of maximal elements and the set of quasi pseudo-complementations on L, provided there is a quasi pseudo-complementation on L. We introduce the concept of $*-$congruence, kernel ideals on a quasi-p-ADL and give equivalent conditions under which every ideal of L is a kernel ideal.

2. Preliminaries

In this section, we give the definition and some elementary properties of a pseudo-complemented ADL and Stone ADL [9, 10]. For the concept of ADL refer to [11] and for the concept of minimal prime ideals in an ADL refer to [6].

Definition 2.1. Let $(L, \vee, \wedge)$ be an ADL with 0. Then a unary operation $a \mapsto a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$ it satisfies the following conditions:

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$,

(2) $a \wedge a^* = 0$,

(3) $(a \vee b)^* = a^* \wedge b^*$.

L is called a Stone ADL if, for any $x \in L$, $x^* \vee x^{**} = 0^*$.

If $(L, \vee, \wedge)$ is an ADL with 0 and $*$ is a pseudo-complementation on L, then we say that $(L, \vee, \wedge, *, 0)$ is a pseudo-complemented ADL (p-ADL, for brevity).

In the following, we give an example of an ADL with a pseudo-complementation which is not a Lattice.

Example 2.2. Let $(X, \vee, \wedge, 0)$ be a discrete ADL. Fix $x_0 \neq 0$ in X and define $*$ on X as follows

$$a^* = \begin{cases} 0, & \text{if } a \neq 0; \\ x_0, & \text{if } a = 0. \end{cases}$$

Then $*$ is a pseudo-complementation on X.

Now we give some elementary properties of pseudo-complementation.

Theorem 2.3. *Let L be an ADL with 0 and $*$ a pseudo-complementation on L and $a, b \in L$. Then we have the following:*

(1) 0^* *is maximal element,*

(2) $0^{**} = 0$,

(3) $a^{**} \wedge a = a$,

(4) $a^{***} = a^*$,

(5) $a^* \wedge b^* = b^* \wedge a^*$,

(6) $a \leq b \Rightarrow b^* \leq a^*$,

(7) $a^* \leq (a \wedge b)^*$ *and* $b^* \leq (a \wedge b)^*$,

(8) $a \wedge b = 0 \Leftrightarrow a^{**} \wedge b = 0$,

(9) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

Definition 2.4. For any non-empty subset A of an ADL L with 0, define

$$A^* = \{x \in L \mid x \wedge a = 0, \text{ for all } a \in A\}.$$

This A^* is an ideal of L and is called the annihilator ideal of A.

For any $a \in L$, we write $[a]^*$ for $\{a\}^*$ and is called annulet of L.
It can be easily observed that, for any subset A of L, $A \cap A^* = \{0\}$.

Lemma 2.5. *Let L be an ADL with 0 and $a \in L$. Then $(a] = L$ if and only if a is a maximal element.*

Theorem 2.6. *Let L be an ADL with 0. Then for any $a \in L$, the annulet $[a]^*$ is a principal ideal if and only if L has a pseudocomplementation.*

Theorem 2.7. *Let L be an ADL with 0 and $*$ a pseudo-complementation on L. For any $a^*, b^* \in L^*$, define $a^* \leq b^*$ if and only if $a^* \wedge b^* = a^*$. Then $(L^*, \leq)$ is a Boolean algebra.*

Corollary 2.8. *Let L be an ADL with 0 and $*$ a pseudo-complementation on L. Then the map $f : L \mapsto L^*$ defined by $f(a) = a^{**}$ is an epimorphism.*

Theorem 2.9. *Let I be an ideal of L and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal (filter) P of L such that $I \subseteq P$ and $P \cap F = \emptyset$ ($F \subseteq P$ and $P \cap I = \emptyset$).*

3. Quasi pseudo-complementation on an ADL

In this section, we give the definition of a quasi pseudo-complementation on an ADL with 0 and study some elementary properties of quasi pseudo-complementation.

Definition 3.1. Let $(L, \vee, \wedge)$ be an ADL with 0. Then a unary operation $a \mapsto a^*$ on L is called a quasi pseudo-complementation on L if, for any $a, b \in L$, the following are satisfied

(1) 0^* is a maximal element,

(2) $(a \vee b)^* = a^* \wedge b^*$,

(3) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,

(4) $a^{***} = a^*$,

(5) $a \wedge a^* = 0$.

If $(L, \vee, \wedge)$ is an ADL with 0 and $*$ a quasi pseudo-complementation on L then we say that $(L, \vee, \wedge, *, 0)$ is a quasi pseudo-complemented ADL. For brevity, we will call quasi pseudo-complemented ADL as q-p-ADL.

Note that every p-ADL is a q-p-ADL but converse need not be true which we show in the following example.

Example 3.2. (i) Let $L = \{0, a, b, c\}$. Define two binary operations $\vee$ and $\wedge$ on L as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

and define $x^* = 0$ for all $x \neq 0$ and $0^* = a$. Then $(L, \vee, \wedge, 0)$ is a distributive lattice and hence an ADL and $*$ is a quasi pseudo-complementation on L but not a pseudo-complementation on L. We can observe that $b \wedge c = 0$ but $b^* \wedge c = 0 \wedge c = 0 \neq c$.

(ii) Let $D = \{0', a', b'\}$ be a discrete ADL and $L = \{0, a, b, c\}$ a distributive lattice given in Example 3.2(i). Then

$$R = D \times L = \big\{(0',0),(0',a),(0',b),(0',c),(a',0),(a',a),(a',b),(a',c),(b',0),\\ (b',a),(b',b),(b',c)\big\}$$

and hence $(R, \vee, \wedge, 0^\diamond)$ is an ADL which is not a lattice, where $0^\diamond = (0', 0)$, under point-wise operation. Define $(x, y)^* = (0', 0)$ for all $(x, y) \neq (0', 0)$ and $(0', 0)^* = (a', a)$. Then $*$ is a quasi pseudo-complementation on R. But it is not a pseudo-complementation on R because $(0', b) \wedge (0', c) = (0', b \wedge c) = (0', 0)$ implies that $(0', b)^* \wedge (0', c) = (0', 0) \wedge (0', c) = (0', 0) \neq (0', c)$.

Example 3.3. Let $(L, +, \cdot, 0)$ be a commutative regular ring. To each $a \in L$, let a° be the unique idempotent element in L such that $aL = a^\circ L$. Define, for any $a, b \in L$,

(i) $a \wedge b = a^\circ b$,

(ii) $a \vee b = a + (1 - a^\circ)b$,

(iii) $a^* = 1 - a^\circ$,

then $(L, \vee, \wedge, 0)$ is an almost distributive lattice with 0 and $*$ is a quasi pseudo-complementation on L.

Now we give some elementary properties of a quasi pseudo-complementation.

Lemma 3.4. *Let L be an ADL with 0 and $*$ a quasi pseudo-complementation on L. Then, for $a, b \in L$, we have the following:*

(1) $a^* \wedge a = 0$,

(2) $0^{**} = 0$,

(3) $a^* \wedge b^* = b^* \wedge a^*$,

(4) $a^* \wedge a^{**} = 0$,

(5) $a \leq b \Rightarrow b^* \leq a^*$,

(6) $a \wedge b^* \leq a \wedge (a \wedge b)^*$,

(7) $(a \vee b)^* = (b \vee a)^*$,

(8) $(a \wedge b)^* = (b \wedge a)^*$,

(9) $a^* \wedge (a^* \wedge b)^* = a^* \wedge b^*$,

(10) $a \wedge b^* = 0 \Rightarrow a^* \wedge b^* = b^*$ *and* $a^{**} \wedge b^{**} = a^{**}$.

Proof. (1) $a^* \wedge a = a \wedge a^* \wedge a = 0 \wedge a = 0$.

(2) Since 0^* is a maximal element, we have $0^* \vee 0 = 0^*$. So that $0^{**} = (0^* \vee 0)^* = 0^{**} \wedge 0^* = 0$.

(3) We know that for any $a, b \in L$, $a \vee 0 = a$ and $b \vee 0 = b$. Therefore $a^* \wedge 0^* = a^*$ and $b^* \wedge 0^* = b^*$. Then $a^* \leq 0^*$ and $b^* \leq 0^*$ and hence $a^* \wedge b^* = b^* \wedge a^*$.

(4) Since $a \wedge a^* = 0$, we have $(a \wedge a^*)^{**} = 0^{**} = 0$. Hence, by Definition 3.1(3, 4), $a^{**} \wedge a^* = 0$. Thus $a^* \wedge a^{**} = a^{**} \wedge a^* \wedge a^{**} = 0$.

(5) Suppose $a \leq b$. Then $a \vee b = b$. So that $b^* = (a \vee b)^* = a^* \wedge b^* = b^* \wedge a^*$ by (3). Hence $b^* \leq a^*$.

(6) Since $a \wedge b \leq b$, by (4), we get $b^* \leq (a \wedge b)^*$ and hence $a \wedge b^* \leq a \wedge (a \wedge b)^*$

(7) $(a \vee b)^* = a^* \wedge b^* = b^* \wedge a^* = (b \vee a)^*$.

(8) $(a \wedge b)^* = (a \wedge b)^{***} = (a^{**} \wedge b^{**})^* = (b^{**} \wedge a^{**})^* = (b \wedge a)^{***} = (b \wedge a)^*$.

(9) $a^* \wedge (a^* \wedge b)^* = [a \vee (a^* \wedge b)]^* = [(a \vee a^*) \wedge (a \vee b)]^{***} = [(a \vee a^*)^{**} \wedge (a \vee b)^{**}]^* = [0^* \wedge (a \vee b)^{**}]^* = (a \vee b)^{***} = (a \vee b)^* = a^* \wedge b^*$.

(10) Suppose $a \wedge b^* = 0$. Then $b^* = 0^* \wedge b^* = (a \wedge b^*)^* \wedge b^* = b^* \wedge (b^* \wedge a)^* = b^* \wedge a^*$. So that $b^* \leq a^*$ and hence $a^{**} \leq b^{**}$. Therefore $a^{**} \wedge b^{**} = a^{**}$. ∎

Now we prove that quasi-pseudo-complementation on an ADL is equationally definable.

Theorem 3.5. *Let* L *be an ADL with* 0. *Then* $*$ *is a quasi pseudo-complementation on* L *if and only if*

(1) $(a \wedge b)^* = (a \wedge b^{**})^*$

(2) 0^* *is a maximal element*

(3) $(a \vee b)^* = a^* \wedge b^*$

(4) $(a \wedge b)^* = (b \wedge a)^*$

(5) $a \wedge a^* = 0$.

Proof. Suppose $*$ is a quasi pseudo-complementation on L and $a, b \in L$. Then $(2), (3), (4)$ and (5) follow from Definition 3.1 and Lemma 3.4. Now

$$\begin{aligned} (a \wedge b)^* &= (a \wedge b)^{***} \\ &= (a^{**} \wedge b^{**})^* \\ &= (a^{**} \wedge b^{****})^* \\ &= (a \wedge b^{**})^{***} \\ &= (a \wedge b^{**})^*. \end{aligned}$$

Conversely, assume that the conditions hold. Let $a, b \in L$. Then

$$a^* = (0^* \wedge a)^* = (0^* \wedge a^{**})^* = a^{***}$$

and

$$\begin{aligned} (a \wedge b)^{**} &= ((a \wedge b)^*)^* \\ &= ((a \wedge b^{**})^*)^* \\ &= ((a^{**} \wedge b^{**})^*)^* \\ &= (a^* \vee b^*)^{***} \\ &= (a^* \vee b^*)^* \\ &= a^{**} \wedge b^{**}. \end{aligned}$$

■

Now we give necessary and sufficient conditions for a q-p-ADL to be a p-ADL.

Theorem 3.6. *Let L be an ADL with 0 and $*$ is a quasi pseudo-complementation on L. Then, for $a, b \in L$, the following are equivalent*

(1) $*$ *is a pseudo-complementation on* L

(2) $a^{**} \wedge a = a$

(3) $a^* \wedge b = (a \wedge b)^* \wedge b$

(4) $[a]^* \subseteq (a^*]$.

Proof. (1)⇒(2) is clear.

(2)⇒(1): Assume (2). Let $a, b \in L$ and $a \wedge b = 0$. Then

$$\begin{aligned} b &= b^{**} \wedge b \quad \text{(by (2))} \\ &= 0^* \wedge b^{**} \wedge b \\ &= (a^* \wedge a^{**})^* \wedge b^{**} \wedge b \\ &= (a \vee a^*)^{**} \wedge b^{**} \wedge b \\ &= b^{**} \wedge (a \vee a^*)^{**} \wedge b \\ &= (b \wedge (a \vee a^*))^{**} \wedge b \\ &= [(b \wedge a) \vee (b \wedge a^*)]^{**} \wedge b \\ &= [0 \vee (b \wedge a^*)]^{**} \wedge b \\ &= (b \wedge a^*)^{**} \wedge b \\ &= b^{**} \wedge a^{***} \wedge b \\ &= a^{***} \wedge b^{**} \wedge b \\ &= a^* \wedge b. \end{aligned}$$

Therefore $*$ is a pseudo-complementation on L. Similarly, we can prove (1) $\Leftrightarrow$ (3) and (1) $\Leftrightarrow$ (4). ■

Now, we prove that if $*$ is a quasi pseudo-complementation on an ADL L, the set $S(L) = \{a^* \mid a \in L\} = \{a \in L \mid a = a^{**}\}$ becomes a Boolean algebra.

Theorem 3.7. *Let L be an ADL with 0 and $*$ a quasi pseudo-complementation on L. For any $a^*, b^* \in S(L)$, define $a^* \leq b^*$ if and only if $a^* \wedge b^* = a^*$. Then $(S(L), \leq)$ is a Boolean algebra.*

Proof. Clearly $\leq$ is a partial ordering on $S(L)$. Let $a^*, b^* \in S(L)$. Since $(a \vee b)^* = a^* \wedge b^*$, we have $a^* \wedge b^* \in S(L)$ and $a^* \wedge b^* = b^* \wedge a^*$. So that $a^* \wedge b^*$ is the greatest lower bound of $\{a^*, b^*\}$ in $S(L)$. Now we prove that $(a^{**} \wedge b^{**})^*$ is the lub of a^*, b^* in the poset $(S(L), \leq)$. We have $a^{**} \wedge b^{**} \leq b^{**}$ and hence $b^* = b^{***} \leq (a^{**} \wedge b^{**})^*$. Similarly, we get that $a^* \leq (a^{**} \wedge b^{**})^*$. Therefore $(a^{**} \wedge b^{**})^*$ is an upper bound of $\{a^*, b^*\}$ in $S(L)$. Let $c^* \in S(L)$ and $a^* \leq c^*$, $b^* \leq c^*$. Then $c^{**} \leq a^{**}$ and $c^{**} \leq b^{**}$. Hence $c^{**} \leq a^{**} \wedge b^{**}$. Therefore $(a^{**} \wedge b^{**})^* \leq c^{***} = c^*$. Thus $(a^{**} \wedge b^{**})^*$ is the least upper bound of $\{a^*, b^*\}$ in $S(L)$ and we denote this by $a^* \underline{\vee}\, b^*$. Hence $(S(L), \leq)$ is a lattice.

It can be easily seen that $(S(L), \leq)$ is a bounded lattice in which 0^* is the greatest element and 0 is the least element. Let $a^* \in S(L)$. Then $a^{**} \in S(L)$, $a^* \underline{\vee}\, a^{**} = (a^{**} \wedge a^{***})^* = 0^*$ and $a^* \wedge a^{**} = 0$. Hence a^{**} is the complement of a^* in $S(L)$. Finally we prove that $S(L)$ is distributive. Let a^*, b^* and $c^* \in S(L)$. Then,

$$\begin{aligned} a^* \underline{\vee}\, (b^* \wedge c^*) &= [a^{**} \wedge (b^* \wedge c^*)^*]^* \\ &= [a^{****} \wedge (b^{**} \vee c^{**})^{**}]^* \text{ by definition 3.1} \\ &= [a^{**} \wedge (b^{**} \vee c^{**})]^{***} \text{ by definition 3.1} \\ &= [a^{**} \wedge (b^{**} \vee c^{**})]^* \text{ by definition 3.1} \\ &= [(a^{**} \wedge b^{**}) \vee (a^{**} \wedge c^{**})]^* \\ &= (a^{**} \wedge b^{**})^* \wedge (a^{**} \wedge c^{**})^* \\ &= (a^* \underline{\vee}\, b^*) \wedge (a^* \underline{\vee}\, c^*). \end{aligned}$$

Therefore $a^* \underline{\vee}\, (b^* \wedge c^*) = (a^* \underline{\vee}\, b^*) \wedge (a^* \underline{\vee}\, c^*)$. Thus $(S(L), \leq)$ is a Boolean algebra. ■

Corollary 3.8. *Let L be an ADL with 0 and $*$ a quasi pseudo-complementation on L. Then the map $f : L \mapsto S(L)$ defined by $f(a) = a^{**}$ is an epimorphism.*

Definition 3.9. Two quasi pseudo-complementations $*$ and $\perp$ on an ADL L are said to be equivalent, denoted by $* \approx \perp$, if $0^* = 0^{\perp}$. Then clearly $\approx$ is an equivalence relation on the set $\mathcal{QPC}(L)$, of all quasi pseudo-complementations on L.

Theorem 3.10. *Let* $(L, \vee, \wedge, 0)$ *be an ADL with a quasi pseudo-complementation* $*$ *and* M *the set of all maximal elements in* L. *Then, for any* $m \in M$, $*_m : L \times L \to L$ *defined by* $a^{*_m} = a^* \wedge m$ *for all* $a \in L$ *is again a quasi pseudo-complementation on* L *and the correspondence* $m \mapsto *_m$ *induces a bijection of* M *onto* $\mathcal{QPC}(L)/\approx$.

Proof. Let $a, b \in L$ and $m \in M$. Then we can easily show that $*_m$ is a quasi pseudo-complementation on L. Let $m, n \in M$ such that $*_m \approx *_n$. Then $0^{*_m} = 0^{*_n}$ which implies that $0^* \wedge m = 0^* \wedge n$ and hence $m = n$ since 0^* is maximal in L. Now, let $\perp \in \mathcal{QPC}(L)$. Then $m = 0^\perp \in M$ and $0^\perp = 0^* \wedge 0^\perp = 0^* \wedge m = 0^{*_m}$ and hence $*_m \approx \perp$. Thus $m \mapsto *_m$ is a bijection of M onto $\mathcal{QPC}(L)/\approx$. ■

Now we give some equivalent conditions for a q-p-ADL to be a Stone ADL.

Theorem 3.11. *Let* L *be a q-p-ADL. Then the following are equivalent.*

(i) *L is a Stone ADL.*

(ii) $a^* \vee a^{**} = 0^*$ *for all* $a \in L$.

Proof. (i)⇒(ii) is clear. Assume (ii). Let $a \in L$. Then $a^* \vee a^{**} = 0^*$ implies that $(a^* \vee a^{**}) \wedge a = 0^* \wedge a$ which gives $a^{**} \wedge a = a$. Hence, by Theorem 3.6, L is pseudo-complemented and hence L is a Stone ADL. ■

Theorem 3.12. *Let* L *be a q-p-ADL. Then the following are equivalent.*

(i) *L is a Stone ADL.*

(ii) *For any* $a, b \in L, (a \wedge b)^* = a^* \vee b^*$.

Proof. Assume (i). Suppose $a, b \in L$ and $x = (a \wedge b)^*$. Then $a \wedge b \wedge x = 0$ implies that $a^* \wedge b \wedge x = b \wedge x$ which gives $a^{**} \wedge b \wedge x = 0$. So that $b^* \wedge a^{**} \wedge x = a^{**} \wedge x$ and hence $b^* \vee (a^{**} \wedge x) = b^*$. Now, $a^* \vee b^* = a^* \vee [b^* \vee (a^{**} \wedge x)] = a^* \vee (b^* \vee x)$. Thus $(a^* \vee b^*) \wedge x = [a^* \vee (b^* \vee x)] \wedge x = x$. Now $(a \wedge b)^* = (a^* \vee b^*) \wedge (a \wedge b)^* = [a^* \wedge (a \wedge b)^*] \vee [b^* \wedge (a \wedge b)^*] = a^* \vee b^*$. Conversely, assume (ii). Let $a \in L$. Then $a^* \vee a^{**} = (a \wedge a^*)^* = 0^*$. Hence, by Theorem 3.11, (i) follows. ■

There are no hidden difficulties to prove the following theorem. Hence we omit its proof.

Theorem 3.13. *Let* L *be a q-p-ADL. Then the following are equivalent.*

(i) *L is a Stone ADL,*

(ii) $S(L)$ *is a sublattice of* L,

(iii) $(a \vee b)^{**} = a^{**} \vee b^{**}$ *for all* $a, b \in L$,

(iv) $a \wedge b = 0$ *implies* $a^* \vee b^* = 0^*$ *for all* $a, b \in L$.

Definition 3.14. Let L be an ADL with 0. An element b in L is said to be a semi-complement of the element a in L if $a \wedge b = 0$. We denote the set of all semi-complements of a by $S(a)$.

Lemma 3.15. *Let L be an ADL with $a \in L$. Then $S(a)$ is an ideal of L.*

Lemma 3.16. *Let L be a q-p-ADL. Then the following are equivalent.*

(i) *L is a p-ADL.*

(ii) *$S(a) = (a^*]$ for all $a \in L$.*

Definition 3.17. An ideal I of an ADL L is called a direct factor if there exists an ideal J of L such that $I \cap J = \{0\}$ and $I \vee J = L$.

Now we prove the following.

Theorem 3.18. *Let L be a q-p-ADL. Then L is a Stone ADL if and only if, for any $a \in L$, the ideal $S(a) = (a^*]$ is a direct factor of L.*

Proof. Suppose L is a Stone ADL and $a \in L$. Then $a^* \vee a^{**} = 0^*$ and $S(a) = (a^*]$. Now $a^* \wedge a^{**} = 0$ and $a^* \vee a^{**} = 0^*$ implies that $(a^*] \wedge (a^{**}] = (0]$ and $(a^*] \vee (a^{**}] = L$. Hence $(a^*]$ is a direct factor of L. Conversely, assume that $S(a) = (a^*]$ is a direct factor of L, for all $a \in L$. Then there exists an ideal J in L such that $(a^*] \cap J = \{0\}$ and $(a^*] \vee J = L$. Write $0^* = b \vee (a^* \wedge x)$ for some $x \in L, b \in J$. Also $a^* \wedge b \in (a^*] \wedge J$ which implies that $a^{**} \wedge b = b$ and $a^{**} \vee b = b$. Now, $0^* = (a^{**} \wedge 0^*) \vee 0^* = (a^{**} \wedge 0^*) \vee ((a^* \vee b) \wedge 0^*) = (a^{**} \vee a^* \vee b) \wedge 0^* = (a^* \vee a^{**}) \wedge 0^*$. Hence $0^* = (a^* \wedge 0^*) \vee (a^{**} \wedge 0^*) = (a \vee 0)^* \vee (a^* \vee 0)^* = a^* \vee a^{**}$. Thus L is a Stone ADL. ■

4. Kernel ideals in q-p-ADLs

In this section, we introduce the notions of $*-$congruences and kernel ideals on a q-p-ADL L. We give a necessary and sufficient condition for a congruence on L to be a $*-$congruence and we characterize kernel ideals. Finally we give equivalent conditions for every ideal of L to become a kernel ideal. We can recall that a congruence relation on an ADL $(L, \vee, \wedge, 0)$ is an equivalence relation θ, compatible with the operations $\vee$ and $\wedge$. Throughout this section, L stands for a q-p-ADL $(L, \vee, \wedge, 0)$ with quasi pseudo-complementation $*$, otherwise we specify.

Definition 4.1. A congruence relation θ on a q-p-ADL L is called a $*-$congruence if it satisfies the following condition:

$$(a, b) \in \theta \text{ implies that } (a^*, b^*) \in \theta \text{ for all } a, b \in L.$$

The following example shows that every congruence on a q-p-ADL need not be a $*-$congruence.

Example 4.2. Let $R = D \times L = \{(0', 0), (0', a), (0', b), (0', c), (a', 0), (a', a), (a', b), (a', c), (b', 0), (b', a), (b', b), (b', c)\}$ be a q-p-ADL as in Example 3.2(ii). Now consider two congruence relations θ_1 and θ_2 on $R = D \times L$ whose partitions A_1 and A_2 are respectively given by

$$A_1 = \Big\{\{(0', 0), (0', a), (0', b), (0', c), (a', a)\}, \{(a', 0), (a', b), (a', c)\}, \{(b', 0), (b', a), (b', b), (b', c)\}\Big\}$$

and

$$A_2 = \Big\{\{(0', 0), (0', a), (0', b), (0', c)\}, \{(a', 0), (a', a), (a', b), (a', c)\}, \{(b', 0), (b', a), (b', b), (b', c)\}\Big\}.$$

Then clearly θ_1 is a $*-$congruence on $R = D \times L$. But θ_2 is not a $*-$congruence on $R = D \times L$, because $((0', b), (0', 0)) \in \theta_2$ and $((0', b)^*, (0', 0)^*) = ((0', 0), (a', a)) \notin \theta_2$.

Now we give an equivalent condition for a congruence relation on q-p-ADL L to be $*-$congruence.

Theorem 4.3. *A congruence relation θ on L is a $*-$congruence if and only if $(a, 0) \in \theta$ implies that $(a^*, 0^*) \in \theta$ for any $a \in L$.*

Proof. Let θ be a $*-$congruence on L and $a \in L$. Then $(a, 0) \in \theta$ implies $(a^*, 0^*) \in \theta$. Conversely, assume that the condition holds and $(a, b) \in \theta$. Then $(b, a) \in \theta$ which implies that $(b \wedge a^*, 0) \in \theta$ and hence $((b \wedge a^*)^*, 0^*) \in \theta$. Therefore $(a^* \wedge b^*, a^*) = (a^* \wedge (a^* \wedge b)^*, a^* \wedge 0^*) \in \theta$. Similarly, we can obtain that $(a^* \wedge b^*, b^*) \in \theta$. Hence $(a^*, b^*) \in \theta$. Thus θ is a $*-$congruence on L. ■

We proved that $S(L) = \{x \in L \mid x^{**} = x\}$ is a Boolean algebra in which for any $a, b \in S(L), a \underline{\vee} b = (a^* \wedge b^*)^*$. In a pseudo-complemented distributive lattice, the relation θ defined by $(x, y) \in \theta$ if and only if $x^* = y^*$ is a congruence called the Glivenko congruence. Now, we prove that the same θ is a $*-$congruence relation on a q-p-ADL L and we show that L/θ is a Boolean algebra under this $*-$congruence θ on L

Theorem 4.4. *Let L be a q-p-ADL. Then L/θ is a Boolean algebra under the $*-$congruence relation θ on L defined by $(x, y) \in \theta$ if and only if $x^* = y^*$.*

Proof. Clearly θ is an equivalence relation on L. Suppose $(a,b) \in \theta$ and $c \in L$. Then $a^* = b^*$ and hence $(a \vee c)^* = a^* \wedge c^* = b^* \wedge c^* = (b \vee c)^*$. Again $(a \wedge c)^* = (a^{**} \wedge c^{**})^* = (b^{**} \wedge c^{**})^* = (b \wedge c)^*$. Hence $(a \vee c, b \vee c) \in \theta$ and $(a \wedge c, b \wedge c) \in \theta$. Then θ is a congruence relation on L. Clearly θ is a $*-$congruence. Now define $\lambda : L/\theta \to S(L)$ by $\lambda([a]_\theta) = a^{**}$ for all $[a]_\theta \in L/\theta$. Clearly λ is well-defined, one-one and onto. Let $[a]_\theta, [b]_\theta \in L/\theta$. Now $\lambda([a]_\theta \wedge [b]_\theta) = \lambda([a \wedge b]_\theta) = (x \wedge y)^{**} = x^{**} \wedge y^{**} = \lambda([a]_\theta) \wedge \lambda([b]_\theta)$. Again, $\lambda([a]_\theta \underline{\vee} [b]_\theta) = \lambda([a \vee b]_\theta) = (a \vee b)^{**} = (a^* \wedge b^*)^* = a^{**} \underline{\vee} b^{**} = \lambda([a]_\theta) \underline{\vee} \lambda([b]_\theta)$. Therefore λ is an isomorphism. Hence L/θ is a Boolean algebra. ■

For any ideal I of L, we introduce a $*-$congruence $\psi(I)$ on L corresponding to I.

Theorem 4.5. *Let L be a q-p-ADL and I an ideal of L. Define a binary relation $\psi(I)$ on L by*

$$(a,b) \in \psi(I) \textit{ if and only if } a \wedge i^* = b \wedge i^* \textit{ for some } i \in I.$$

Then $\psi(I)$ is a $*-$congruence relation on L.

Proof. Since $(i \vee j)^* = i^* \wedge j^*$ for any $i, j \in L$ and the fact that I is an ideal of L, clearly $\psi(I)$ is an equivalence relation on L. Let $(a,b) \in \psi(I)$ and $(c,d) \in \psi(I)$. Then $a \wedge i^* = b \wedge i^*$ for some $i \in I$ and $c \wedge j^* = d \wedge j^*$ for some $j \in I$. Hence $(a \vee c) \wedge (i \vee j)^* = (a \vee c) \wedge i^* \wedge j^* = (a \wedge i^* \wedge j^*) \vee (b \wedge i^* \wedge j^*) = (c \wedge i^* \wedge j^*) \vee (d \wedge i^* \wedge j^*)$. Therefore $(a \vee c, b \vee d) \in \psi(I)$. Now $(a \wedge c) \wedge (i \vee j)^* = a \wedge c \wedge i^* \wedge j^* = a \wedge i^* \wedge c \wedge j^* = b \wedge i^* \wedge d \wedge j^*$. Hence $(a \wedge c, b \wedge d) \in \psi(I)$. Thus $\psi(I)$ is a congruence on L. Suppose $(a, 0) \in \phi(I)$. Then $a \wedge i^* = 0$ for some $i \in I$. Then $0^* \wedge i^* = (a \wedge i^*)^* \wedge i^* = a^* \wedge i^*$(by Lemma 3.4(9)). Therefore $(a^*, 0^*) \in \psi(I)$. Thus $\psi(I)$ is a $*-$congruence relation on L. ■

Definition 4.6. An ideal I of an q-p-ADL is called a kernel ideal if there exists a $*-$congruence μ on L such that $I = Ker\mu = \{a \in L \ : \ (a,0) \in \mu\}$.

Theorem 4.7. *If I is a kernel ideal of L then the following conditions hold.*

(i) *$a, b \in I$ implies $(a^* \wedge b^*)^* \in I$.*

(ii) *$a, b \in I$ implies that there exists $k \in I$ such that $a^* \wedge b^* = k^*$.*

Proof. Let I be kernel ideal of L and $a, b \in I$. Then $I = ker\theta$ for some $*-$congruence θ on L. Then $(a, 0) \in \theta$ and $(b, 0) \in \theta$. Hence $(a^*, 0^*) \in \theta$ and $(b^*, 0^*) \in \theta$. So that $(a^* \wedge b^*, 0^*) \in \theta$ and hence $((a^* \wedge b^*)^*, 0) \in \theta$. Thus $(a^* \wedge b^*)^* \in ker\theta = I$. Hence (i) follows. Put $k = (a^* \wedge b^*)^*$. Then, by (i), $k \in I$ and $k^* = (a^* \wedge b^*)^{**} = a^* \wedge b^*$. Hence (ii) follows. ■

Now we give necessary and sufficient conditions for an ideal to become a kernel ideal.

Theorem 4.8. *For any ideal I of L, the following are equivalent.*

(i) *I is a kernel ideal.*

(ii) *For $a, b \in L, a^* = b^*$ and $a \in I$ imply $b \in I$.*

(iii) *$a \in I$ if and only if $a^{**} \in I$.*

Proof. (i)⇒(ii): Assume (i). Then there exists a $*-$congruence θ on L such that $ker\theta = I$. Chose $x, y \in L$ such that $x^* = y^*$ and $x \in I$. Then $(x, 0) \in \theta$ and hence $(y^*, 0^*) = (x^*, 0^*) \in \theta$. Therefore $(0, y) = (y^* \wedge y, 0^* \wedge y) \in \theta$. Thus $y \in ker\theta = I$. Since $x^* = x^{***}$ for all $x \in L$, (ii)⇒(iii) follows. Now, assume (iii). We know that $\psi(I)$ is a $*-$congruence relation on L by Theorem 4.5. If $x \in ker\psi(I)$. Then $(x, 0) \in \psi(I)$ and hence $x \wedge i^* = 0$ for some $i \in I$. Therefore, by Theorem 3.4(10), $x^{**} = x^{**} \wedge i^{**} \in I$ and hence $x \in I$. Thus I is a kernel ideal. ∎

An element $a \in L$ is called a dense element if $a^* = 0$. The set $D(L)$ of all dense elements of L forms a filter of L. The following theorem can be proved easily.

Theorem 4.9. *In L, the following conditions hold.*

(i) *$x \vee x^* \in D(L)$ for all $x \in L$.*

(ii) *$D(L)$ is a filter of L.*

(iii) *For any ideal I with $I \cap D(L) = \emptyset$, there exists a minimal prime ideal P such that $I \subseteq P$ and$P \cap D(L) = \emptyset$.*

(iv) *Every proper kernel ideal in contained in a minimal prime ideal.*

Theorem 4.10. *If $(x] = (x^{**}]$ for all $x \in L$, then $(x]$ is a kernel ideal.*

In [11], it is observed that the set $\mathcal{PI}(L)$ of all principal ideals of an ADL L is a distributive lattice with least element $(0]$. Now, we give sufficient condition for $\mathcal{PI}(L)$ to become Boolean algebra.

Theorem 4.11. If $(x] = (y]$ for all $x, y \in D(L)$ then $\mathcal{PI}(L)$ is a Boolean algebra.

Proof. Let $(x] = (y]$ for all $x, y \in D(L)$. Then $\{(x] \mid x \in D(L)\} = \{(d]\}$ for some $x \in L$. Clearly $x \vee x^* \in D(L)$. Hence $(x \vee x^*] = (d]$. For any $(x] \in \mathcal{PI}(L)$, $(x] \subseteq (x \vee x^*] = (d]$. Therefore $(d]$ is the greatest element of $\mathcal{PI}(L)$. Also $(x] \cap (x^*] = (0]$ and $(x] \vee (x^*] = (d]$. Hence $\mathcal{PI}(L)$ is a bounded distributive lattice in which every element is complemented. Thus $\mathcal{PI}(L)$ is a Boolean algebra. ∎

Now, we give equivalent conditions for every ideal of L to become a kernel ideal.

Theorem 4.12. *Let L be a q-p-ADL. Then the following conditions are equivalent.*

(i) *Every ideal is a kernel ideal.*

(ii) *Every prime ideal is a kernel ideal.*

(iii) *For any* $a, b \in L$, $a^* = b^*$ *implies* $(a] = (b]$.

(iv) *Every principal ideal is a kernel ideal.*

Proof. (i)⇒(ii) is clear. Assume (ii) and $a, b \in L$ such that $a^* = b^*$. Suppose $(a] \neq (b]$. Without loss of generality, assume that $(a] \nsubseteq (b]$. Take $\mathfrak{F} = \{J \in \mathcal{I}(L) \mid b \in J \text{ and } a \notin J\}$. Then, by Zorn's lemma, $\mathfrak{F}$ has a maximal element, say P. Chose $r, s \in L$ such that $r \notin P$ and $s \notin P$. Then $P \subset P \vee (r]$ and $P \subset P \vee (s]$. By the maximality of P, we can get $a \in \{P \vee (r]\} \cap \{P \vee (s]\} = P \vee (r \wedge s]$. If $r \wedge s \in P$, then $a \in P$ which is a contradiction. Hence P is prime which is kernel ideal. Now $a^* = b^*$ and $b \in P$ implies that $a \in P$, which is a contradiction. Therefore $(a] = (b]$. Hence (iii) follows. Now, assume (iii) and I is a principal ideal of L. Then $I = (a]$ for some $a \in L$. Let $r, s \in L$ such that $r^* = s^*$ and $r \in (a]$. Then $(r] = (s]$ and $s \in (r] \subseteq (a]$. Hence (iv) follows. Finally, assume (iv) and I is an ideal of L. Let $a \in I$. Then $(a] \subseteq I$ and hence $a^{**} \in I$ since $(a]$ is a kernel ideal. Conversely assume $a^{**} \in I$. Then $(a^{**}] \subseteq I$ and hence $a \in (a^{**}] \subseteq I$ since $(a^{**}]$ is a kernel ideal. Hence I is a kernel ideal of L. ∎

Conclusion and future work

In this paper, we have introduced the concept of quasi-pseudo-complementation on an ADL as a generalization of pseudo-complemenation on an ADL and studied its properties. We have given necessary and sufficient conditions for a q-p-ADL to be a p-ADL and a stone ADL. We proved that if $*$ is a quasi pseudo-complementation on an ADL L then the set $S(L) = \{a^* \mid a \in L\}$ becomes a Boolean algebra. Also, it is observed that, there exists an induced surjective correspondence between the set of maximal elements and the set of quasi pseudo-complementations on L, provided there is a quasi pseudo-complementation. Also, the concept of $*-$congruence, kernel ideals on a q-p-ADL is introduced and given equivalent conditions for every ideal of L to become a kernel ideal.

In our future work, we will introduce the concepts of demi-pseudo-complementation on an ADL(for brevity, demi-p-ADL), Weak-Stone ADL and study their properties.

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