

SYMMETRIC BI-DERIVATIONS IN ALMOST DISTRIBUTIVE LATTICES

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Abstract

In this paper, we introduce the concept of symmetric bi-derivation in an Almost Distributive Lattice (ADL) and derive some important properties of symmetric bi-derivations in ADLs.

Keywords: Almost Distributive Lattice (ADL), derivations, isotone derivations and symmetric bi-derivations.

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1. INTRODUCTION

In 1980, the concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [13]. This class of ADLs include most of the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other.

The concept of derivation in an ADL was introduced in our earlier paper [11]. The notion of derivation in Lattices was first given in Szasz [14] in 1974. Posner [9] introduced derivations in ring theory and later several authors worked on it ([2, 5]). Several authors worked on derivations in Lattices ([1, 3, 4, 6, 7, 8, 14, 15, 16] and [17]). We have introduced the concept of f -derivations in an ADL in our paper [12]. The concept of symmetric bi-derivations in lattices was introduced by Ceven [4] in 2009.

In this paper, we introduce the concept of symmetric bi-derivation in an ADL and investigate some important properties. Also, we define the trace d of a

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symmetric bi-derivation D on an ADL L and prove some important properties based on it. We define a fixed set $Fix_d(L)$ and prove that it is a weak ideal if d is the trace of a join preserving symmetric bi-derivation D on an associative ADL L . Also, we introduce the concept of an isotone symmetric bi-derivation in an ADL and we establish a set of conditions which are sufficient for a symmetric bi-derivation on an ADL with a maximal element to become an isotone symmetric bi-derivation. We prove that if an ADL L has a maximal element, then the trace of a join preserving symmetric bi-derivation on L is a homomorphism. Finally, we prove that the set of all principal symmetric bi-derivations on an ADL L forms an ADL.

2. PRELIMINARIES

In this section, we recollect certain basic concepts and important results on Almost Distributive Lattices.

Definition 2.1 [10]. An algebra (L, \vee, \wedge) of type $(2, 2)$ is called an Almost Distributive Lattice, if it satisfies the following axioms:

$$L_1 : (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) \quad (RD\wedge)$$

$$L_2 : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad (LD\wedge)$$

$$L_3 : (a \vee b) \wedge b = b$$

$$L_4 : (a \vee b) \wedge a = a$$

$$L_5 : a \vee (a \wedge b) = a.$$

Definition 2.2 [10]. Let L be any non-empty set. Define, for any $x, y \in L$, $x \vee y = x$ and $x \wedge y = y$. Then (L, \vee, \wedge) is an ADL and it is called a discrete ADL.

Throughout this paper L stands for an ADL (L, \vee, \wedge) unless otherwise specified.

Lemma 2.3 [10]. For any $a, b \in L$, we have

$$(i) \quad a \wedge a = a,$$

$$(ii) \quad a \vee a = a,$$

$$(iii) \quad (a \wedge b) \vee b = b,$$

$$(iv) \quad a \wedge (a \vee b) = a,$$

$$(v) \quad a \vee (b \wedge a) = a,$$

$$(vi) \quad a \vee b = a \text{ if and only if } a \wedge b = b,$$

$$(vii) \quad a \vee b = b \text{ if and only if } a \wedge b = a.$$

Definition 2.4 [10]. For any $a, b \in L$, we say that a is less than or equal to b and write $a \leq b$, if $a \wedge b = a$ or, equivalently, $a \vee b = b$.

Theorem 2.5 [10]. *For any $a, b, c \in L$, we have the following*

- (i) *The relation \leq is a partial ordering on L ,*
- (ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ (LD \vee),
- (iii) $(a \vee b) \vee a = a \vee b = a \vee (b \vee a)$,
- (iv) $(a \vee b) \wedge c = (b \vee a) \wedge c$,
- (v) *The operation \wedge is associative in L ,*
- (vi) $a \wedge b \wedge c = b \wedge a \wedge c$.

Theorem 2.6 [10]. *For any $a, b \in L$, the following are equivalent.*

- (i) $(a \wedge b) \vee a = a$,
- (ii) $a \wedge (b \vee a) = a$,
- (iii) $(b \wedge a) \vee b = b$,
- (iv) $b \wedge (a \vee b) = b$,
- (v) $a \wedge b = b \wedge a$,
- (vi) $a \vee b = b \vee a$,
- (vii) *The supremum of a and b exists in L and equals to $a \vee b$,*
- (viii) *there exists $x \in L$ such that $a \leq x$ and $b \leq x$,*
- (ix) *the infimum of a and b exists in L and equals to $a \wedge b$.*

Definition 2.7 [10]. L is said to be associative, if the operation \vee in L is associative.

Theorem 2.8 [10]. *The following are equivalent.*

- (i) L is a distributive lattice,
- (ii) *the poset (L, \leq) is directed above,*
- (iii) $a \wedge (b \vee a) = a$, for all $a, b \in L$,
- (iv) *the operation \vee is commutative in L ,*
- (v) *the operation \wedge is commutative in L ,*
- (vi) *the relation $\theta := \{(a, b) \in L \times L \mid a \wedge b = b\}$ is anti-symmetric,*
- (vii) *the relation θ defined in (vi) is a partial order on L .*

Lemma 2.9 [10]. *For any $a, b, c, d \in L$, we have the following*

- (i) $a \wedge b \leq b$ and $a \leq a \vee b$,
- (ii) $a \wedge b = b \wedge a$ whenever $a \leq b$,
- (iii) $[a \vee (b \vee c)] \wedge d = [(a \vee b) \vee c] \wedge d$,
- (iv) $a \leq b$ implies $a \wedge c \leq b \wedge c$, $c \wedge a \leq c \wedge b$ and $c \vee a \leq c \vee b$.

Definition 2.10 [10]. An element $0 \in L$ is called zero element of L , if $0 \wedge a = 0$ for all $a \in L$.

Lemma 2.11 [10]. *If L has 0 , then for any $a, b \in L$, we have the following*

- (i) $a \vee 0 = a$, (ii) $0 \vee a = a$, (iii) $a \wedge 0 = 0$
and
- (iv) $a \wedge b = 0$ if and only if $b \wedge a = 0$.

Definition 2.12 [13]. Let L be a non-empty set and $x_0 \in L$. Define, for $x, y \in L$,
 $x \wedge y = y$ if $x \neq x_0$
 $= x$ if $x = x_0$ and
 $x \vee y = x$ if $x \neq x_0$
 $= y$ if $x = x_0$, then (L, \vee, \wedge, x_0) is an ADL with x_0 as zero element. This is called discrete ADL with zero.

Definition 2.13 [10]. An element $x \in L$ is called maximal if, for any $y \in L$, $x \leq y$ implies $x = y$.

We immediately have the following.

Lemma 2.14 [10]. *For any $m \in L$, the following are equivalent:*

- (1) m is maximal,
- (2) $m \vee x = m$ for all $x \in L$,
- (3) $m \wedge x = x$ for all $x \in L$.

Definition 2.15 [10]. A nonempty subset I of L is said to be an ideal if and only if it satisfies the following:

- (1) $a, b \in I \Rightarrow a \vee b \in I$,
- (2) $a \in I, x \in L \Rightarrow a \wedge x \in I$.

Definition 2.16. A nonempty subset I of L is said to be a weak ideal if and only if it satisfies the following:

- (1) $a, b \in I \Rightarrow a \vee b \in I$,
- (2) $a \in I, x \in L$ and $x \leq a \Rightarrow x \in I$.

Observe that every ideal of L is a weak ideal, but not conversely.

Definition 2.17 [10]. If L_1, L_2 are ADLs, then a function $f : L_1 \rightarrow L_2$ is said to be a homomorphism if it satisfies the following:

- (1) $f(x \wedge y) = fx \wedge fy$,
- (2) $f(x \vee y) = fx \vee fy$ for all $x, y \in L$.

Definition 2.18. A function $d : L \rightarrow L$ is called an isotone, if $dx \leq dy$ for any $x, y \in L$ with $x \leq y$.

3. SYMMETRIC BI-DERIVATIONS IN ADLS

We begin this section with the following definition of a derivation in an ADL.

Definition 3.1 [11]. A function $d : L \rightarrow L$ is called a derivation on L , if $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$ for all $x, y \in L$.

Definition 3.2.

- (i) A mapping $D : L \times L \rightarrow L$ is called symmetric if $D(x, y) = D(y, x)$ for all $x, y \in L$,
- (ii) D is called an isotone map if, for any $x, y, z \in L$ with $x \leq y$, $D(x, z) \leq D(y, z)$.

The following definition introduces the notion of a symmetric bi-derivation on ADLs.

Definition 3.3. A symmetric function $D : L \times L \rightarrow L$ is called a symmetric bi-derivation on L , if $D(x \wedge y, z) = [y \wedge D(x, z)] \vee [x \wedge D(y, z)]$.

Observe that a symmetric bi-derivation D on L also satisfies $D(x, y \wedge z) = [z \wedge D(x, y)] \vee [y \wedge D(x, z)]$ for all $x, y, z \in L$.

Example 3.4. Let L be an ADL and $a \in L$. If we define a mapping $D_a : L \times L \rightarrow L$ by $D_a(x, y) = x \wedge y \wedge a$ for all $x, y \in L$, then D_a is a symmetric bi-derivation on L .

The symmetric bi-derivation D_a given in the above example, is called as the principal symmetric bi-derivation on L induced by $a \in L$.

Example 3.5. Let L be an ADL with 0 and $0 \neq a \in L$. If we define a mapping $D : L \times L \rightarrow L$ by $D(x, y) = (x \vee y) \wedge a$ for all $x, y \in L$, then D is a symmetric map, but not a symmetric bi-derivation on L .

Example 3.6. In a discrete ADL $L = \{0, a, b\}$, if we define $D : L \times L \rightarrow L$ by $D(x, y) = a$ when $(x, y) \in \{(0, 0), (0, a), (a, 0)\}$ and $D(x, y) = 0$ otherwise, then D is a symmetric map but not a symmetric bi-derivation on L .

Example 3.7. Let L be a discrete ADL. Then any constant map $D : L \times L \rightarrow L$ is a symmetric bi-derivation on L .

Definition 3.8. If D is a symmetric bi-derivation on L , then the mapping $d : L \rightarrow L$ defined by $d(x) = D(x, x)$ for all $x \in L$ is called the trace of D .

Theorem 3.9. Let D be a symmetric bi-derivation on L and d be the trace of D . Then the following hold:

1. $D(x, y) = x \wedge D(x, y)$ for all $x, y \in L$,

2. $dx = x \wedge dx$ for any $x \in L$,
3. $D(x, dx) = dx$ for any $x \in L$,
4. $d^2x = dx$ for any $x \in L$.

Proof. (1) Let $x, y \in L$. Then $D(x, y) = D(x \wedge x, y) = [x \wedge D(x, y)] \vee [x \wedge D(x, y)] = x \wedge D(x, y)$.

(2) Let $x \in L$. Then replace y by x in (1) above, we get $D(x, x) = x \wedge D(x, x)$. That is $dx = x \wedge dx$.

(3) Let $x \in L$. Then $D(x, dx) = D(x, x \wedge dx) = [dx \wedge D(x, x)] \vee [x \wedge D(x, dx)] = dx \vee D(x, dx) = dx$.

(4) Let $x \in L$. Then $d^2x = D(dx, dx) = D(x \wedge dx, dx) = [dx \wedge D(x, dx)] \vee [x \wedge D(dx, dx)] = dx \vee [x \wedge d^2x] = (dx \vee x) \wedge (dx \vee d^2x) = (dx \vee x) \wedge dx = dx$. ■

Corollary 3.10. Suppose L has 0 and D is a symmetric bi-derivation on L . Then $D(0, x) = 0$ for all $x \in L$.

Definition 3.11. A symmetric map $D : L \times L \rightarrow L$ is called a joini preserving map if $D(x \vee y, z) = D(x, z) \vee D(y, z)$ for all $x, y, z \in L$.

Let us recall that in an ADL, it is not known whether \vee is associative or not. If \vee is associative in an ADL, then it is called an associative ADL. Now we prove the following.

Lemma 3.12. Let d be the trace of a join preserving symmetric bi-derivation D on an associative ADL L . Then $d(x \vee y) = dx \vee D(x, y) \vee dy$ for all $x, y \in L$ and d is an isotone map on L .

Proof. Let $x, y \in L$. Then $d(x \vee y) = D(x \vee y, x \vee y) = D(x, x) \vee D(x, y) \vee D(y, x) \vee D(y, y) = dx \vee D(x, y) \vee dy$. If $x \leq y$, then $dy = d(x \vee y) = dx \vee D(x, y) \vee dy$. Thus $dx \leq dy$. Hence d is an isotone map on L . ■

Lemma 3.13. Let d be the trace of a symmetric bi-derivation D on an associative ADL L . Then $d(x \wedge y) = (y \wedge dx) \vee D(x, y) \vee (x \wedge dy)$ for all $x, y \in L$.

Proof. Let $x, y \in L$. Then $d(x \wedge y) = D(x \wedge y, x \wedge y) = [y \wedge D(x, x \wedge y)] \vee [x \wedge D(y, x \wedge y)] = [y \wedge ([y \wedge D(x, x)] \vee [x \wedge D(x, y)])] \vee [x \wedge ([y \wedge D(y, x)] \vee [x \wedge D(y, y)])] = (y \wedge dx) \vee D(x, y) \vee D(y, x) \vee (x \wedge dy) = (y \wedge dx) \vee D(x, y) \vee (x \wedge dy)$. ■

Corollary 3.14. If d is the trace of a symmetric bi-derivation D on L , then $y \wedge dx \leq d(x \wedge y)$ for all $x, y \in L$.

Corollary 3.15. Suppose m is a maximal element of L and d is the trace of a symmetric bi-derivation on L . Then, for any $x \in L$, we have,

1. $x \geq dm$ implies $dx \geq dm$,
2. $x \leq dm$ implies $dx = x$.

Proof. (1) If $x \geq dm$, then $dm = (x \wedge dm) \leq d(m \wedge x)$ by above Corollary. Thus $dx \geq dm$.

(2) If $x \leq dm$, then $x = x \wedge dm \leq d(m \wedge x) = dx$. Hence $dx = x \wedge dx = x$. ■

Theorem 3.16. *Let d be the trace of a join preserving symmetric bi-derivation D on an associative ADL L . Then $Fix_d(L) = \{x \in L/dx = x\}$ is a weak ideal of L .*

Proof. Let $x \in L$, $y \in Fix_d(L)$ and $x \leq y$. Then, by Lemma 3.13, $dx = d(y \wedge x) = (x \wedge dy) \vee D(x, y) \vee (y \wedge dx) = (x \wedge y) \vee D(x, y) \vee (y \wedge x \wedge dx) = x \vee D(x, y) \vee (x \wedge dx) = x \vee (x \wedge dx) = x$. Thus $x \in Fix_d(L)$. Now, let $x, y \in Fix_d(L)$. Then, by Lemma 3.12, $d(x \vee y) = dx \vee D(x, y) \vee dy = x \vee D(x, y) \vee y = x \vee y$. Thus $x \vee y \in Fix_d(L)$. Hence $Fix_d(L)$ is a weak ideal of L . ■

Theorem 3.17. *Let m be a maximal element of L and d be the trace of a symmetric bi-derivation D on L . Then the following are equivalent.*

1. d is an isotone map on L ,
2. $dx = x \wedge dm$ for all $x \in L$,
3. $d(x \wedge y) = dx \wedge dy$ for all $x, y \in L$,
4. $d(x \vee y) = dx \vee dy$ for all $x, y \in L$.

Proof. (1) \Rightarrow (2) : Let $x \in L$. By Corollary 3.14, $x \wedge dm \leq d(m \wedge x) = dx$. On the other hand, since d is isotone, $d(x \wedge m) \leq dm$. Thus, by Corollary 3.14, $m \wedge dx \leq d(x \wedge m) \leq dm$. Now, $dx = x \wedge dx = m \wedge x \wedge dx = x \wedge m \wedge dx \leq x \wedge dm$. Hence $dx = x \wedge dm$.

(2) \Rightarrow (3) : Let $x, y \in L$. Then $d(x \wedge y) = x \wedge y \wedge dm = x \wedge dm \wedge y \wedge dm = dx \wedge dy$.

(2) \Rightarrow (4) : Let $x, y \in L$. Then $d(x \vee y) = (x \vee y) \wedge dm = (x \wedge dm) \vee (y \wedge dm) = dx \vee dy$.

(3) \Rightarrow (1) and (4) \Rightarrow (1) are trivial. ■

Theorem 3.18. *Suppose L has a maximal element m . Then the trace of every join preserving symmetric bi-derivation on L is a homomorphism.*

Proof. Let d be the trace of a join preserving symmetric bi-derivation D on L and $x, y \in L$ with $x \leq y$. Then $d(x \vee y) = D(x \vee y, x \vee y) = D(x, x \vee y) \vee D(y, x \vee y) = [D(x, x) \vee D(x, y)] \vee D(y, x \vee y) = [dx \vee D(x, y)] \vee D(y, x \vee y)$. Now, $dx \leq dx \vee D(x, y) \leq d(x \vee y) = dy$. Hence d is an isotone map on L . Therefore, by Theorem 3.17, d is a homomorphism. ■

Finally we conclude this section with the following Theorem.

Theorem 3.19. *Let $\mathcal{D}(L)$ be the set of all principal symmetric bi-derivations on L . Then $(\mathcal{D}(L), \vee, \wedge)$ is an ADL.*

Proof. For $a, b, x, y \in L$, define $(D_a \vee D_b)(x, y) = D_a(x, y) \vee D_b(x, y)$ and $(D_a \wedge D_b)(x, y) = D_a(x, y) \wedge D_b(x, y)$.

Now, $(D_a \vee D_b)(x, y) = D_a(x, y) \vee D_b(x, y) = (x \wedge y \wedge a) \vee (x \wedge y \wedge b) = (x \wedge y) \wedge (a \vee b) = D_{a \vee b}(x, y)$. Thus $D_a \vee D_b = D_{a \vee b} \in \mathcal{D}(L)$. Now $(D_a \wedge D_b)(x, y) = D_a(x, y) \wedge D_b(x, y) = (x \wedge y \wedge a) \wedge (x \wedge y \wedge b) = (x \wedge y) \wedge (a \wedge b) = D_{a \wedge b}(x, y)$. Thus $D_a \wedge D_b = D_{a \wedge b} \in \mathcal{D}(L)$. Therefore, $\mathcal{D}(L)$ is closed under \vee and \wedge and clearly it satisfies the properties of an ADL. ■

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