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A NOTE ON SEMIDIRECT SUM OF LIE ALGEBRAS

Tadeusz Ostrowski

The State Higher Vocational School in Gorzów Wlkp.
Technical Institute
ul. Mysliborska 34, 66–400 Gorzów Wlkp., Poland

e-mail: Tadostr@Interia.pl

Abstract

In the paper there are investigated some properties of Lie algebras, the construction which has a wide range of applications like computer sciences (especially to computer visions), geometry or physics, for example. We concentrate on the semidirect sum of algebras and there are extended some theoretic designs as conditions to be a center, a homomorphism or a derivative. The Killing form of the semidirect sum where the second component is an ideal of the first one is considered as well.

Keywords: Lie algebra, subalgebra, ideal, center, semidirect sum, homomorphism, derivation, Killing form.

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1. Preliminaries

The aim of the paper is to extend some theoretic designs of Lie algebras for the concept of the semidirect sum of Lie algebras. The construction of the semidirect sum is important for both theoretical and practical reasons. It can be viewed as a generalization of the direct sum of Lie algebras. The construction appears in the Levi decomposition (see, for example, [3, 15], or [22] for details) and it is one of the main tools for obtaining so called non-reductive Lie algebras. The Levi decomposition expresses a Lie algebra as a sum of two components, semisimple and solvable. It gives a possibility to reduce problems about finite dimensional Lie algebras (over a field of characteristic zero) by separating them in those two classes. The construction has its applications in physics (it appears in Young-Mills equations [14]) or in geometry (an affine space is the semidirect sum of the general linear Lie algebra and a vector space as an abelian Lie algebra).

Lie algebras can be constructed in many various ways; via linear space or linear maps (endomorphisms) on a vector space, via vector fields, via Lie groups, via a set of structure constants, or via differential operators (see [11] for details). In the paper we shall restrict our attention to the first way. To establish notation recall some definitions.

Definition 1. An algebraic structure $g = (V, [\,,])$ is said to be the Lie algebra, if V is a linear space over a field F, and V is endowed with a multiplication of its elements and a binary operation $[\,,]$ (the Lie bracket or commutator) satisfying the following axioms:

(1a) A.1 (bilinearity): $[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C]$,

(1b) $[A, \alpha B + \beta C] = \alpha[A, B] + \beta[A, C]$,

(2) A.2 (skew-symmetry): $[A, B] + [B, A] = 0$,

(3) A.3 (the Jacobi identity): $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

for all scalars $\alpha, \beta \in F$ and elements $A, B, C \in V$.

Recall that the vector product is not associative but holds the Jacobi identity

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0;\ a, b, c \in R^3$$

and by defining $[a, b] = a \times b$ we obtain the Lie algebra structure on R^3. It is interesting that the geometrical meaning of the Jacobi identity is equivalent to the basic fact that altitudes of a triangle are intersected in the one point [1].

Under assumption $\mathrm{char} F \neq 2$ the axiom (2) can be replaced by $[A, A] = 0$; $A \in \mathbf{g}$. Throughout the paper we shall always take F to be the field of characteristic zero.

It will be convenient to identify $\mathbf{g}$ and V and denote a commutant of sets $\mathbf{a}$ and $\mathbf{b}$ by $[\mathbf{a}, \mathbf{b}]$, i.e., the linear span of all elements of the form $[A, B]$; $A \in \mathbf{a}$, $B \in \mathbf{b}$.

Definition 2. A sub Lie algebra of $\mathbf{g}$ is a subspace $\mathbf{h} \subset \mathbf{g}$ that is closed under the Lie bracket (closed under commutation with itself) i.e.,

$$[\mathbf{h}, \mathbf{h}] \subset \mathbf{h}. \tag{4}$$

Definition 3. An ideal in $\mathbf{g}$ is a subalgebra $\mathbf{a}$ such that $\mathbf{a}$ is invariant by commutation with all elements of $\mathbf{g}$:

$$[\mathbf{a}, \mathbf{g}] \subset \mathbf{a}. \tag{5}$$

The condition (5) is stronger than (4). To be the ideal $\mathbf{a}$ is closed not just under commutation with itself, but also under commutation with any other element of $\mathbf{g}$. It is obvious that, by (2), there is no need to distinguish between left and right ideals.

Definition 4. The center of a Lie algebra $\mathbf{g}$, denoted by $Z(\mathbf{g})$, is a set of elements commuting with $\mathbf{g}$, that is

$$Z(\mathbf{g}) = \{A \in \mathbf{g} : [A, \mathbf{g}] = 0\}. \tag{6}$$

Clearly, $Z(\mathbf{g})$ is an ideal of the Lie algebra $\mathbf{g}$, so it is an important example of an ideal which frequently is not trivial. In literature one can find the equivalent definition of the center (see, for example, [20]), namely $Z(\mathbf{g}) = \max \mathbf{a} : [\mathbf{a}, \mathbf{g}] = \{0\}$.

Definition 5. Let $M_{n,n}(F)$ denotes a linear space of n by n matrices over a field F. A general linear algebra, denoted by $\mathbf{gl}_n(F)$, is the Lie algebra having the structure

$$\mathbf{gl}_n(F) = \{A \in M_{n,n}(F), [\,,]\}. \tag{7}$$

$\mathbf{gl}_n(F)$ is called *classical algebra*, which is also applied to some subalgebras of $\mathbf{gl}_n(F)$, like special linear Lie algebra $\mathbf{sl}_n(F) = \{A \in M_{n,n}(F) : trA = 0\}$, n-th orthogonal Lie algebra $\mathbf{o}_n(F) = \{A \in M_{n,n}(F) : A + A^T = 0\}$ or special orthogonal Lie algebra $\mathbf{so}_n(F) = \{A \in \mathbf{sl}_n(F) : A + A^T = 0\}$. For classical Lie algebras it is known $[\mathbf{sl}_n(F), \mathbf{gl}_n(F)] \subset \mathbf{sl}_n(F)$, $Z(\mathbf{gl}_n(F)) = \{\alpha I : \alpha \in F\}$, $Z(\mathbf{sl}_n(F)) = \{0\}$, $Z(\mathbf{so}_n(F)) = \mathbf{so}_n(F)$; $n = 2$, $Z(\mathbf{so}_n(F)) = \{0\}$; $n \neq 2$.

Matrix Lie algebras are especially important, because, according to deep Ado's theorem, every finite-dimensional real or complex Lie algebra has a faithful representation by matrices (the theorem is proved, for instance, in [22]).

Definition 6. Let $\mathbf{g}$ be any algebra over a field F. A derivation of $\mathbf{g}$ (the Lie derivative) is a linear operator δ on $\mathbf{g}$ satisfying the Leibniz law with respect to the commutator:

$$\partial([A, B]) = [\partial(A), B] + [A, \partial(B)];\ A, B \in \mathbf{g}. \tag{8}$$

Definition 7. For any element $A \in \mathbf{g}$, an inner derivation (also referred as adjoint homomorphism, adjoint representation or the adjoint of A) of the Lie algebra $\mathbf{g}$ is the operator ad_A defined as follows

$$\mathrm{ad}_A(B) = [A, B],\ B \in \mathbf{g}. \tag{9}$$

It is clear that the inner derivation holds the following properties:

(i) $\mathrm{ad}_C(AB) = \mathrm{ad}_C(A)B + A\mathrm{ad}_C(B)$,

(ii) $\mathrm{ad}_A[B,C] = [\mathrm{ad}_A(B), C] + [B, \mathrm{ad}_A(C)]$,

(iii) $\mathrm{ad}_{[A,B]} = [\mathrm{ad}_A, \mathrm{ad}_B]$,

(iv) $[\mathbf{g},\mathbf{g}] \subset Z(\mathbf{g}) \Longleftrightarrow \mathrm{ad}_{A}\circ \mathrm{ad}_B = \mathrm{ad}_{[A,B]}$.

Properties (ii) and (iii) create another forms of Jacobi identity (see [12]; (iii) is proved in [9]), expressed in the term of the adjoint homomorphism (thinking of $[X, Y]$ as a linear map in Y for each fixed X gives a somewhat different perspective [9]). An ideal $\mathbf{a}$ in $\mathbf{g}$ can be defined with the adjoint homomorphism as a subset for which $\mathbf{g}$ is invariant under the adjoint representation, i.e., $\mathrm{ad}_a(\mathbf{g}) \subset \mathbf{a}$. Similarly, (6) can be rewritten as $Z(\mathbf{g}) = \mathrm{Ker}(\mathrm{ad}_g)$.

The inner derivation is a very important case of derivations because if a Lie algebra is semisimple (the algebra has no non-zero abelian ideals) then every derivation is inner (for proof see, for instance, [18]).

Definition 8. The set $\mathrm{Der}(\mathbf{g})$ of linear operators on $\mathbf{g}$ which are also derivations of $\mathbf{g}$ is called the derivation algebra.

For example, $\mathbf{sl}_n(F) = \mathrm{Der}(\mathbf{gl}_n(F))$. It is obvious, that ad_g is an ideal of $\mathrm{Der}(\mathbf{g})$, i.e.,

$$[\mathrm{ad}_g, \mathrm{Der}(\mathbf{g})] \subset \mathrm{ad}_g. \tag{10}$$

Definition 9. Let $\mathbf{g}$ and $\mathbf{h}$ be two Lie algebras over the same ground field F. Let be given a linear map $f : \mathbf{g} \to \mathbf{h}$ that is compatible with the commutators. Saying more precisely, for all elements A, B in $\mathbf{g}$, the following equality holds

$$f([A,B]) = [f(A), f(B)]. \tag{11}$$

The map f is called a homomorphism (or a Lie algebra morphism) between $\mathbf{g}$ and $\mathbf{h}$.

Clearly, $\mathrm{Ker}f$ is an ideal in $\mathbf{g}$, and $\mathrm{Im}f$ is the Lie subalgebra of $\mathbf{h}$ (proof can be found, for example, in [5]). Formally

$$[\mathrm{Ker}f, \mathbf{g}] \subset \mathrm{Ker}f, [\mathrm{Im}f, \mathrm{Im}f] \subset \mathrm{Im}f.$$

A simple example of a Lie algebra morphism is $\mathrm{Tr} : \mathbf{gl}_n(F) \to F$. Of course, the trace is linear and (11) holds. Moreover, KerTr$=\mathbf{sl}_n(F)$, and the first isomorphism theorem implies that $\mathbf{gl}_n(F)/\mathbf{sl}_n(F) \cong F$.

Definition 10. Lie algebra is called solvable if the Lie algebra commutator series (called the derived series) $\mathbf{g}^i = [\mathbf{g}^{i-1}, \mathbf{g}^{i-1}]$, $i = 1, 2, \ldots$, vanishes for some i.

2. Semidirect sum (product) of Lie algebras

2.1. The construction

Definition 11. Let $\mathbf{g}$ and $\mathbf{h}$ be given Lie algebras. A semidirect sum (product) of $\mathbf{g}$ and $\mathbf{h}$ is the vector space $\mathbf{g} \times \mathbf{h}$ made into a Lie algebra by defining

$$(12) \quad [(A,X),(B,Y)]_\tau = ([A,B],[X,Y]+\tau(A)Y-\tau(B)X); \ A,B \in \mathbf{g}, \ X,Y \in \mathbf{h},$$

where $\tau \in \mathrm{Hom}(\mathbf{g}, \mathrm{Der}\mathbf{h})$. The semidirect sum will be denoted here by $\mathbf{g} \oplus_\tau \mathbf{h}$.

In a particular case, when $\tau(\mathbf{g})\mathbf{h} = 0$, then the semidirect sum reduces to the direct sum of Lie algebras, where

$$[(A,X),(B,Y)] = ([A,B],[X,Y]); \ A,B \in \mathbf{g}, X,Y \in \mathbf{h}.$$

Example 1. Let V be a module or a vector space over a field F and $\dim V = n$. The structure $(V,[\,,])$ is a commutative Lie algebra, that is $[\mathbf{u},\mathbf{v}] = 0$; $\mathbf{u},\mathbf{v} \in V$. Together with the general linear Lie algebra can be constructed the semidirect sum $\mathbf{gl}_n(F) \oplus_\tau V$, where $\tau \in \mathrm{Hom}(\mathbf{gl}_n(F), \mathrm{Der}V)$ is the induced action of $\mathbf{gl}_n(F)$ on V by concatenation:

$$\tau(A)\mathbf{v} = A\mathbf{v}; \ A \in \mathbf{gl}_n(F), \ \mathbf{v} \in V.$$

It is easy to see that for all $A \in \mathbf{gl}_n n(F)$, $\mathbf{u},\mathbf{v} \in V$ the Leibniz condition (8) is fulfilled by identity, that is

$$A[\mathbf{u},\mathbf{v}] = [A\mathbf{u},\mathbf{v}] + [\mathbf{u},A\mathbf{v}],$$

which can be rewritten as

$$\tau(A)[\mathbf{u},\mathbf{v}] = [\tau(A)\mathbf{u},\mathbf{v}] + [\mathbf{u},\tau(A)\mathbf{v}].$$

The equality above means that the algebra $\mathbf{gl}_n(F)$ stands for the derivation algebra of the Lie algebra $(V,[\,,])$. Formally, $\mathbf{gl}_n(F) = \mathrm{Der}V$.

The commutator of $\mathbf{gl}_n(F) \oplus_\tau V$ is given as follows:

$$[(A,\mathbf{u}),(B,\mathbf{v})]_\tau = ([A,B], A\mathbf{v} - B\mathbf{u}), \text{ for all } A,B \in \mathbf{gl}_n(F), \ \mathbf{u},\mathbf{v} \in V.$$

Multiplication of elements of the semidirect product is defined as follows:

$$(A,\mathbf{u})(B,\mathbf{v}) = (AB, A\mathbf{v} + \mathbf{u}) \text{ for all } A \in \mathbf{gl}_n(F), \mathbf{u},\mathbf{v} \in V.$$

The unity of multiplication is the element $(I,0)$, and if A is nonsingular then

$$(A,\mathbf{u})^1 = (A^1, -A^1\mathbf{u}).$$

Example 2. Euclidean algebra $\mathbf{e}(3)$ of the form

$$\mathbf{e}(3) = \{ \begin{bmatrix} 0 & -c & b & d \\ c & 0 & -a & e \\ -b & a & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix} ; [\,,]\}$$

which stands for the Lie algebra of the group of isometries of R^3 is the semidirect sum of the angular momentum algebra and abelian algebra R^3 :

$$\mathbf{e}(3) = \mathbf{so}_3(R) \oplus_\tau R^3,$$

where by isomorphisms R^3 is identified with an algebra of $R^{4,4}$ matrices, and $\mathbf{so}_3(R)$ is identified with the algebra of $R^{4,4}$ matrices and with vectors from R^3 as well, that is

$$\{\mathbf{u} \in R^3 : \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\} \cong \{X \in R^{4,4} : X = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} ; x, y, z \in R\},$$

$$\mathbf{so}_3(R) = \{ \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \} \cong \{A : A = \begin{bmatrix} 0 & -c & b & 0 \\ c & 0 & -a & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\},$$

and A is identified with the vector of the form $[a\ b\ c]^T$.

Therefore $\tau(A)X = AX$ means the cross product of two vectors from R^3 via their matrix representations in $R^{4,4}$. If

$$A = \begin{bmatrix} 0 & -c & b & 0 \\ c & 0 & -a & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -f & e & 0 \\ f & 0 & -d & 0 \\ -e & d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then, of course, $[A, B]$ can be also identified with

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} bf - ce \\ cd - af \\ ae - bd \end{bmatrix}.$$

It is easy to see that $\tau : \mathbf{so}_3(R) \to \operatorname{Der} R^3$ and for any $A, B \in \mathbf{so}_3(R)$, $\mathbf{u}, \mathbf{v} \in R^3$

$$\tau(A)[\mathbf{u}, \mathbf{v}] = [\tau(A)\mathbf{u}, \mathbf{v}] + [\mathbf{u}, \tau(A)\mathbf{v}]$$

and, by definition, the commutator of the semidirect sum has the form

$$[(A,\mathbf{u}),(B,\mathbf{v})]_\tau = ([A,B], A\mathbf{v} - B\mathbf{u}).$$

For more examples of the semidirect sums of Lie algebras see [21]. A generalization of the construction $\mathbf{gl}_n(F) \oplus_\tau V$ can be found in [10, 13].

The semidirect sum of the form $\mathbf{g} \oplus_\tau \mathbf{a}$, where $\mathbf{a}$ is a commutative ideal has been researched by Panyushev in [16]. Note that if $\tau = \mathrm{ad}$, then the commutator of the semidirect sum $\mathbf{g} \oplus_{\mathrm{ad}} \mathbf{a}$ (there is no restriction for the ideal $\mathbf{a}$) has the form

$$[(A,B),(X,Y)]_{\mathrm{ad}} = ([A,X], [B,Y] + [A,Y] - [X,B]). \tag{13}$$

Some results connected with the semidirect sum of a semisimple Lie algebra and solvable Lie algebra can be found in [4]. For some special Lie algebras the semidirect sum (called the wreath product) has been recently investigated by a number of researchers; Sushchansky, Netreba [19], Petrogradsky, Razmyslov, Shishkin [17].

2.2. Some properties of the semidirect sum

Let be given a semidirect sum of Lie algebras $\mathbf{g} \oplus_\tau \mathbf{h}$. It is well known that the algebra $\mathbf{g}$, identified by isomorphism with $\mathbf{g} \oplus_\tau O$, is an ideal of the semidirect sum iff the semidirect sum reduces to the direct sum, and the second algebra $\mathbf{h}$, identified by isomorphism with $O \oplus_\tau \mathbf{h}$, is always an ideal of the semidirect sum, that is

$$\text{(i)}\quad [\mathbf{g} \times O,\, \mathbf{g} \times \mathbf{h}]_\tau \subset \oplus_\tau O \Longleftrightarrow \tau(\mathbf{g})\mathbf{h} = O, \tag{14}$$

$$\text{(ii)}\quad [O \times \mathbf{h},\, \mathbf{g} \times \mathbf{h}]_\tau \subset O \oplus_\tau \mathbf{h}. \tag{15}$$

A proof can be found in [8]. Now we can go to some results of the paper. At first will be given a necessary condition for the direct sum of centers of Lie algebras to be the center of the semidirect sums of those algebras.

Theorem 1. *Let be given a semidirect sum $\mathbf{g} \oplus_\tau \mathbf{h}$ of Lie algebras $\mathbf{g}$ and $\mathbf{h}$. Then the following relations hold*

$$Z(\mathbf{g} \oplus_\tau \mathbf{h}) \subset Z(\mathbf{g}) \times \operatorname{Ker} \tau(\mathbf{g}), \tag{16}$$

$$(A,B) \in Z(\mathbf{g} \oplus_\tau \mathbf{h}) \Rightarrow \tau(A) = -\mathrm{ad}_B. \tag{17}$$

Proof. Let fix any $(A,B) \in Z(\mathbf{g} \oplus_\tau \mathbf{h})$. For all elements $(X,O) \in \mathbf{g} \times \mathbf{h}$, we have then

$$[(A,B),(X,O)]_\tau = ([A,X], \tau(X)B) = (O,O).$$

Therefore

$$[A,X]=O\Rightarrow A\in Z(\mathbf{g}),\ \text{ and }\tau(X)B=O\Rightarrow B\in \operatorname{Ker}\tau(\mathbf{g}),$$

On the other hand, for all elements (O,Y) we obtain

$$[(A,B),(O,Y)]_\tau=(O,[B,Y]+\tau(A)Y)=(O,O),$$

which implies that $\tau(A)=-\mathrm{ad}_B$. ∎

It is easy to see that in any semidirect sum $\mathbf{g}\oplus_\tau\mathbf{h}$ of Lie algebras, for $\tau:\mathbf{g}\to \mathrm{Der}\mathbf{h}$ the following relations hold:

$$\tau\in \mathrm{Hom}(\mathbf{g},\mathrm{Der}\mathbf{h})\Rightarrow \tau(Z(\mathbf{g}))\subset Z(\mathrm{Der}\mathbf{g}),$$

$$\tau\in \mathrm{Isom}(\mathbf{g},\mathrm{Der}\mathbf{h})\Rightarrow \tau(Z(\mathbf{g}))= Z(\mathrm{Der}\mathbf{g}).$$

2.3. Derivation, homomorphism, Killing form

At first it will be given a necessary and sufficient condition for the product of derivatives of two Lie algebras to be a derivative of the semidirect sums of the algebras.

Theorem 2. *Let $\mathbf{g}_i$ $(i=1,2)$ be Lie algebras and let $\partial_i\in\mathrm{Der}\mathbf{g}_i$ $(i=1,2)$. Then the following equivalence holds true*

$$\partial_1\times\partial_2\in \mathrm{Der}(\mathbf{g}_1\oplus_\tau\mathbf{g}_2)\Longleftrightarrow(\tau\circ\partial_1)(A)=[\partial_2,\tau(A)],A\in\mathbf{g}_1. \tag{18}$$

Proof. $\Rightarrow$ If $\partial_1\times\partial_2\in \mathrm{Der}(\mathbf{g}_1\otimes_\tau\mathbf{g}_2)$, then for all $P,Q\in\mathbf{g}_1\times\mathbf{g}_2$, by (8), we have

$$(\partial_1\times\partial_2)[P,Q]_\tau=[(\partial_1\times\partial_2)P,Q]_\tau+[P,(\partial_1\times\partial_2)Q]_\tau. \tag{19}$$

Since any element $(A,B)\in\mathbf{g}_1\times\mathbf{g}_2$ can be decomposed as the sum

$$(A,B)=(A,O)+(O,B),$$

then on the left side of (19), for $P=(A,O),Q=(O,B)$, applying linearity of the derivations and bilinearity of the commutator, we obtain

$$\begin{aligned}&(\partial_1\times\partial_2)[(A,O),(O,B)]_\tau\\&=(\partial_1\times\partial_2)([A,O],[O,B]+\tau(A)B-\tau(O)O)=(\partial_1\times\partial_2)(O,\tau(A)B)\\&=(O,(\partial_2\circ\tau(A))B).\end{aligned}$$

On the right side of (19), by (17) and (9), we have as follows

$$\begin{aligned}
&[(\partial_1 \times \partial_2)(A, O), (O, B)]_\tau + [(A, O), (\partial_1 \times \partial_2)(O, B)]_\tau \\
&= [(\partial_1 A, O), (O, B)]_\tau + [(A, O), (O, \partial_2 B)]_\tau \\
&= (O, \tau(\partial_1(A)B) + (O, (\tau(A) \circ \partial_2)B) = (O, (\tau \circ \partial_1(A) + \tau(A) \circ \partial_2)B).
\end{aligned}$$

Therefore $\partial_2 \circ \tau(A) = \tau \circ \partial_1(A) + \tau(A) \circ \partial_2$, and from above

$$(\tau \circ \partial_1)(A) = \partial_2 \circ \tau(A) - \tau(A) \circ \partial_2 = [\partial_2, \tau(A)].$$

$\Leftarrow$ Let $P = (A, B), Q = (X, Y) \in \mathbf{g}_1 \otimes_\tau \mathbf{g}_2$.

$$\begin{aligned}
&[(\partial_1 \times \partial_2)P, Q]_\tau + [P, (\partial_1 \times \partial_2)Q]_\tau \\
&= [(\partial_1 \times \partial_2)(A, B), (X, Y)]_\tau + [(A, B), (\partial_1 \times \partial_2)(X, Y)]_\tau \\
&= [(\partial_1 A, \partial_2 B), (X, Y)]_\tau + [(A, B), (\partial_1 X, \partial_2 Y)]_\tau \\
&= ([\partial_1 A, X], [\partial_2 B, Y] + \tau(\partial_1 A)Y - \tau(X)\partial_2 B) \\
&\quad + ([A, \partial_1 X], [B, \partial_2 Y] + \tau(A)\partial_2 Y - \tau(\partial_1 X)B) \\
&= (\partial_1[A, X], \partial_2[B, Y] + (\partial_2 \circ \tau(A))Y - (\partial_2 \circ \tau(X))B] \\
&= (\partial_1 \times \partial_2)([A, X], [B, Y] + \tau(A)Y - \tau(X)B] \\
&= (\partial_1 \times \partial_2)[(A, B), (X, Y)]_\tau = (\partial_1 \times \partial_2)[P, Q]_\tau.
\end{aligned}$$

■

Next theorem expresses a necessary and sufficient condition for the product of homomorphisms of Lie algebras to be a homomorphism of their semidirect sum.

Theorem 3. *Let* $\mathbf{g}_1 \oplus_\sigma \mathbf{g}_2, \mathbf{h}_1 \oplus_\tau \mathbf{h}_2$ *be given two semidirect sums of Lie algebras and let* $f_i \in \mathrm{Hom}(\mathbf{g}_i, \mathbf{h}_i)$; $i = 1, 2$. *Then for all* $A \in \mathbf{g}_1$ *the following equivalence holds true*

$$f_1 \times f_2 \in \mathrm{Hom}(\mathbf{g}_1 \oplus_\sigma \mathbf{g}_2, \mathbf{h}_1 \oplus_\tau \mathbf{h}_2) \Longleftrightarrow f_2 \circ \sigma(A) = \tau(f_1(A)) \circ f_2. \tag{20}$$

Proof. $\Leftarrow$ By supposition $f_i \in \mathrm{Hom}(\mathbf{g}_i, \mathbf{h}_i)$; $i = 1, 2$, and linearity of $f_1 \times f_2$ is obvious.

Let $(A, B), (X, Y) \in \mathbf{g}_1 \times \mathbf{g}_2)$. Then, by supposition, the following equalities hold:

$$f_1[A, X] = [f_1(A), f_1(X)], f_2[B, Y] = [f_2(B), f_2(Y)].$$

Therefore, by (12) and by the assumption again, we have

$$
\begin{aligned}
&[(f_1 \times f_2)(A,B),(f_1 \times f_2)(X,Y)]_\tau = [(f_1(A),f_2(B)),(f_1(X),f_2(Y))]_\tau \\
&= ([f_1(A),f_1(X)],[f_2(B),f_2(Y)] + (\tau(f_1(A)) \circ f_2)Y - (\tau(f_1(X)) \circ f_2)B) \\
&= (f_1[A,X], f_2[B,Y] + (f_2 \circ \sigma(A))Y - (f_2 \circ \sigma(X))B) \\
&= (f_1 \times f_2)([A,X],[B,Y] + \sigma(A)Y - \sigma(X)B) = (f_1 \times f_2)[(A,B),(X,Y)]_\sigma .
\end{aligned}
$$

$\Rightarrow$ To prove the necessity condition of (20) let notice first that the mapping $f_1 \times f_2$ preserves commutator, i.e., for any elements $K, L \in \mathbf{g}_1 \times \mathbf{g}_2$

$$(f_1 \times f_2)[K,L]_\sigma = [(f_1 \times f_2)(K),(f_1 \times f_2)(L)]_\tau .$$

Let set then $K = (A,O), L = (O,B)$. Therefore we obtain

$$(f_1 \times f_2)[(A,O),(O,B)]_\sigma = (f_1 \times f_2)(O,\sigma(A)B) = (O,(f_2 \circ \sigma(A)B).$$

On the other hand, we have

$$[(f_1{\times}f_2)(A,O),(f_1{\times}f_2)(O,B)]_\tau = [(f_1A,O),(O,f_2B)]_\tau = (O,(\tau(f_1(A)){\circ}f_2)(B)).$$

From above the following equality holds

$$(O,(f_2 \circ \sigma(A)B) = (O,(\tau(f_1(A) \circ f_2)(B)),$$

which implies $f_2 \circ \sigma(A) = \tau(f_1(A)) \circ f_2$. ■

Recall now that the Killing form on a given Lie algebra $\mathbf{g}$ over a field F is a bilinear symmetric mapping $B : \mathbf{g} \times \mathbf{g} \to F$ defined as $B(A,B) = \mathrm{Tr}(\mathrm{ad}_A \circ \mathrm{ad}_B)$; $A, B \in \mathbf{g}$.

Similarly, on the direct or semidirect sum of Lie algebras $\mathbf{g}$ and $\mathbf{h}$ the Killing form is defined by $B((A,B),(X,Y)) = \mathrm{Tr}(\mathrm{ad}_{(A,B)} \circ \mathrm{ad}_{(X,Y)})$; $(A,B),(X,Y) \in \mathbf{g} \times \mathbf{h}$.

Theorem 4. *Let $\mathbf{g} \oplus_{\mathrm{ad}} \mathbf{a}$ be a semidirect sum, where $\mathbf{a}$ is an ideal of a Lie algebra $\mathbf{g}$ over a field F. Then the following equality holds:*

$$B((A,B),(X,Y)) = B(A,X) + B(A+B,X+Y);\ A,X \in \mathbf{g},\ B,Y \in \mathbf{a}.$$

Proof.

$$(\mathrm{ad}_{(A,B)} \circ \mathrm{ad}_{(X,Y)})(U,V) = [(A,B),[(X,Y)),(U,V)]_{\mathrm{ad}}]_{\mathrm{ad}}$$
$$= [(A,B),([X,U],[Y,V]+[X,V]+[Y,U])]_{\mathrm{ad}}$$
$$= ([A,[X,U]],[A+B,[X+Y,V]+[Y,U]]+[B,[X,U]])$$
$$= (O,[A+B,[Y,U]]+[B,[X,U]]) + ([A,[X,U]],[A+B,[X+Y,V]])$$
$$= (O,((\mathrm{ad}_A+\mathrm{ad}_B)\circ \mathrm{ad}_Y + \mathrm{ad}_B \circ \mathrm{ad}_X)U) + (\mathrm{ad}_A \circ \mathrm{ad}_X)$$
$$\times (\mathrm{ad}_A+\mathrm{ad}_B)\circ(\mathrm{ad}_X+\mathrm{ad}_Y)(U,V).$$

It can be written in the following matrix form

$$\left(\mathrm{ad}_{(A,B)} \circ \mathrm{ad}_{(X,Y)}\right) = \begin{bmatrix} U \\ V \end{bmatrix}$$
$$= \begin{bmatrix} \mathrm{ad}_A \circ \mathrm{ad}_X & O \\ (\mathrm{ad}_A+\mathrm{ad}_B)\circ \mathrm{ad}_Y + \mathrm{ad}_B \circ \mathrm{ad}_X & (\mathrm{ad}_A+\mathrm{ad}_B)\circ(\mathrm{ad}_X+\mathrm{ad}_Y) \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}.$$

By linearity of the adjoint homorphism and bilinearity of the Killing form we have

$$B((A,B),(X,Y)) = \mathrm{Tr}(\mathrm{ad}_A \circ \mathrm{ad}_X) + \mathrm{Tr}((\mathrm{ad}_A+\mathrm{ad}_B)\circ(\mathrm{ad}_X+\mathrm{ad}_Y))$$
$$= B(A,X) + B(A+B,X+Y).$$

■

Corollary. *If* **a** *is an abelian ideal in a Lie algebra* **g** *over a field of characteristic different from 2, then* $B((A,B),(X,Y)) = 2B(A,X)$.

Proof. Really, in this special case we obtain

$$(\mathrm{ad}_{(A,B)} \circ \mathrm{ad}_{(X,Y)})(U,V) = [(A,B),[(X,Y)),(U,V)]_{\mathrm{ad}}]_{\mathrm{ad}}$$
$$= [(A,B),([X,U],[X,V]+[Y,U])]_{\mathrm{ad}}$$
$$= ([A,[X,U]],[A,[X,V]+[Y,U]]+[B,[X,U]])$$
$$= (O,[A,[Y,U]]+[B,[X,U]]) + ([A,[X,U]],[A,[X,V])$$
$$= (O,(\mathrm{ad}_A \circ \mathrm{ad}_Y + \mathrm{ad}_B \circ \mathrm{ad}_X)U) + ((\mathrm{ad}_A \circ \mathrm{ad}_X)\times(\mathrm{ad}_A \circ \mathrm{ad}_X))(U,V).$$

Therefore we can write

$$(\mathrm{ad}_{(A,B)} \circ \mathrm{ad}_{(X,Y)}) = \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \mathrm{ad}_A \circ \mathrm{ad}_X & O \\ \mathrm{ad}_A \circ \mathrm{ad}_Y + \mathrm{ad}_B \circ \mathrm{ad}_X & \mathrm{ad}_A \circ \mathrm{ad}_X \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$$

and from above we obtain $\mathrm{Tr}(\mathrm{ad}_{(A,B)} \circ \mathrm{ad}_{(X,Y)}) = 2\mathrm{Tr}(\mathrm{ad}_A \circ \mathrm{ad}_X)$. ■

3. Some remarks and corollaries

Remark 1. With two Lie algebras $\mathbf{g}, \mathbf{h}$ and with $\tau \in \mathrm{Hom}(\mathbf{g}, \mathrm{Der}\mathbf{h})$ a Lie algebra $\mathbf{f}$ can be constructed on the Cartesian product $\mathbf{g} \times \mathbf{h}$ (Definition 11). The new algebra will be a direct sum or semidirect one. The two possibilities are as follows:

1. $\mathrm{Ker}\tau = \mathbf{g} \Rightarrow \mathbf{f} = \mathbf{g} \oplus \mathbf{h}$, and

$$[(A, X), (B, Y)] = ([A, B], [X, Y]); A, B \in \mathbf{g}, X, Y \in \mathbf{h}.$$

2. $\mathrm{Ker}\tau \neq \mathbf{g} \Rightarrow \mathbf{f} = \mathbf{g} \oplus_\tau \mathbf{h}$, and

$$[(A, X), (B, Y)] = ([A, B], [X, Y] + \tau(A)Y - \tau(B)X).$$

With Levi theorem (called also Levi-Maltsev theorem) we go from the opposite side. The theorem states that any Lie algebra $\mathbf{f}$ is formed as a sum of two components, $\mathbf{g}$ and $\mathbf{h}$, where $\mathbf{g}$ stands for a semisimple Lie algebra, called the Levi factor, and $\mathbf{h}$ is the maximal solvable ideal, called the radical. Since then the subalgebra $\mathbf{g}$ can act on the ideal $\mathbf{h}$, and it leads to two cases:

1. $[\mathbf{g}, \mathbf{h}] = 0 \Rightarrow \mathbf{f} = \mathbf{g} \oplus \mathbf{h}$.
2. $[\mathbf{g}, \mathbf{h}] \neq 0$. It implies the existence of a representation τ of the subalgebra $\mathbf{h}$.

Therefore $\mathrm{ad}_A(X) = \tau(A)X$ for any $A \in \mathbf{g}, X \in \mathbf{h}$, and $\mathbf{f} = \mathbf{g} \oplus_\tau \mathbf{h}$.

Corollary 1. *Theorem* 1 *implies that if* $\mathbf{g} = \mathrm{Ker}\tau$, *then*

$$Z(\mathbf{g} \oplus_\tau \mathbf{h}) = Z(\mathbf{g}) \cap \mathrm{Ker}\tau \times Z(\mathbf{h}) \cap \mathrm{Ker}\tau(\mathbf{g}),$$

or, in other words,

$$(A, B) \in Z(\mathbf{g} \oplus_\tau \mathbf{h}) \Leftrightarrow \begin{cases} A \in Z(\mathbf{g}) \textit{ and } \tau(A)\mathbf{h} = O, \\ B \in Z(\mathbf{g}) \textit{ and } \tau(\mathbf{g})B = O. \end{cases}$$

Remark 2. The semidirect sum of Lie algebras of the form $\mathrm{Ker}\tau \oplus_\tau \mathbf{h}$ is Akivis algebra. In non-associative theory Akivis algebras play the role of Lie algebras in associative theory. Recall that Akivis algebras can be constructed in the following way (see [2, 7] for more details): a vector space V over a field F we endow with two operations - a bilinear commutator [,] and a trilinear associator [, ,], where the following relations are true for all $A, B, C \in V$:

(i) $[A, B] = ABBA$,

(ii) $[A, B, C] = (AB)CA(BC)$,

(iii) $[[A, B], C] + [[B, C], A] + [[C, A], B] =$
$= [A, B, C] + [B, C, A] + [C, A, B] - [A, C, B] - [B, A, C] - [C, B, A]$.

The relation (iii) is called Akivis identity and reduces to Jacobi identity for associative algebras. Therefore Akivis algebra is a generalization of Lie algebra.

Corollary 2. *Notice that Theorem* 2 *implies the following property:*

$$\mathrm{ad}_X \times \tau(X) \in \mathrm{Der}(\mathbf{g} \oplus_\tau \mathbf{h}), \;\; X \in \mathbf{g}. \tag{21}$$

In fact, the property above follows immediately from (18) *by setting*

$$\partial_1 = \mathrm{ad}_x, \partial_2 = \tau(X); \;\; X \in \mathbf{g}.$$

Corollary 3. *It is easy to see that in a particular case of the Theorem* 3, *that is if* $\mathbf{g}_1 = \mathbf{g}_2$, $\mathbf{h}_1 = \mathbf{h}_2$, $f_1 = f_2$, *and* $\tau = \sigma = \mathrm{ad}$, *then* (20) *reduces to* (11).

Remark 3. It is well known that for any Lie algebra $\mathbf{g}$ the inner derivations form an ideal in $\mathrm{Der}(\mathbf{g})$, i.e.,

$$\partial \in \mathrm{Der}(\mathbf{g}) \Rightarrow [\partial, \mathrm{ad}_A] = \mathrm{ad}_{\partial(A)}; \; A \in \mathbf{g}. \tag{22}$$

Analogically is for the semidirect sum of Lie algebras. The following lemma states a connection between the homomorphism τ and the adjoint representation on the second summand of the semidirect sum.

Lemma. *For the semidirect sum* $\mathbf{g} \oplus_\tau \mathbf{h}$ *of Lie algebras the following equality holds*

$$[\tau(A), \mathrm{ad}_B] = \mathrm{ad}_{\tau(A)B}; \; A \in \mathbf{g}, \;\; B \in \mathbf{h}. \tag{23}$$

Proof. Let $Y \in \mathbf{h}$. Since $\tau(A) \in \mathrm{Der}\mathbf{h}$, therefore we have

$$[\tau(A), \mathrm{ad}_B](Y) = \tau(A)[B, Y] - \mathrm{ad}_B(\tau(A)Y]$$
$$= [\tau(A)B, Y] + [B, \tau(A)Y] - [B, \tau(A)Y] = [\tau(A)B, Y] = \mathrm{ad}_{\tau(A)B}(Y).$$

∎

Remark 4. Engels theorem (see [6, 12], or [18] for proof) states that a finite-dimensional Lie algebra $\mathbf{g}$ is nilpotent (the lower central series, $\mathbf{g}_i = [\mathbf{g}, \mathbf{g}_{i-1}]$, $i = 1, 2, \dots$, ends in the zero subspace) iff ad_g is the set of nilpotent operators. It implies that if $\mathbf{g}$ is nilpotent and $\mathbf{h}$ is a subalgebra of $\mathbf{g}$, then the semidirect sum $\mathbf{g} \oplus_{\mathrm{ad}} \mathbf{h}$ has to be nilpotent too. Since any nilpotent Lie algebra is solvable, then $\mathbf{g} \oplus_{\mathrm{ad}} \mathbf{h}$ is solvable as well.

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