\textbf{Abstract}

We call the digraph $D$ an \textit{m-coloured} digraph if its arcs are coloured with \( m \) colours. If $D$ is an \( m \)-coloured digraph and $a \in A(D)$, \textit{colour}($a$) will denote the colour has been used on $a$. A path (or a cycle) is called \textit{monochromatic} if all of its arcs are coloured alike. A $\gamma$-cycle in $D$ is a sequence of vertices, say $\gamma = (u_0, u_1, \ldots, u_n)$, such that $u_i \neq u_j$ if $i \neq j$ and for every $i \in \{0, 1, \ldots, n\}$ there is a $u_iu_{i+1}$-monochromatic path in $D$ and there is no $u_{i+1}u_i$-monochromatic path in $D$ (the indices of the vertices will be taken mod $n+1$). A set $N \subseteq V(D)$ is said to be a \textit{kernel by monochromatic paths} if it satisfies the following two conditions: (i) for every pair of different vertices $u, v \in N$ there is no monochromatic path between them and; (ii) for every vertex $x \in V(D) \setminus N$ there is a vertex $y \in N$ such that there is an $xy$-monochromatic path.

Let $D$ be a finite \( m \)-coloured digraph. Suppose that $\{C_1, C_2\}$ is a partition of $C$, the set of colours of $D$, and $D_i$ will be the spanning subdigraph of $D$ such that $A(D_i) = \{ a \in A(D) \mid \text{colour}(a) \in C_i \}$. In this paper, we give some sufficient conditions for the existence of a kernel by monochromatic paths in a digraph with the structure mentioned above. In particular
we obtain an extension of the original result by B. Sands, N. Sauer and R. Woodrow that asserts: Every 2-coloured digraph has a kernel by monochromatic paths. Also, we extend other results obtained before where it is proved that under some conditions an \(m\)-coloured digraph has no \(\gamma\)-cycles.

**Keywords:** digraph, kernel, kernel by monochromatic paths, \(\gamma\)-cycle.

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1. Introduction

For general concepts we refer the reader to [1, 2]. Let \(D\) be a digraph, and let \(V(D)\) and \(A(D)\) denote the sets of vertices and arcs of \(D\), respectively. We recall that a subdigraph \(D_1\) of \(D\) is a spanning subdigraph if \(V(D_1) = V(D)\). If \(S\) is a nonempty subset of \(V(D)\) then the subdigraph of \(D\) induced by \(S\), denoted by \(D[S]\), is the digraph where \(V(D[S]) = S\) and whose arcs are all those arcs of \(D\) joining vertices of \(S\). An arc \(u_1u_2\) of \(D\) will be called an \(S_1S_2\)-arc of \(D\) whenever \(u_1 \in S_1\) and \(u_2 \in S_2\).

A set \(I \subseteq V(D)\) is independent if \(A(D[I]) = \emptyset\). A kernel \(N\) of \(D\) is an independent set of vertices such that for each \(z \in V(D) \setminus N\) there is an \(zN\)-arc in \(D\), that is an arc from \(z\) toward some vertex in \(N\). A digraph \(D\) is called a kernel perfect digraph when every induced subdigraph of \(D\) has a kernel. Sufficient conditions for the existence of kernels in a digraph have been investigated by several authors, Von Neumann and Morgenstern [26]; Duchet and Meyniel [8]; Duchet [6, 7] and Galeana-Sánchez and Neumann-Lara [14, 15]. The concept of kernel has found many applications, see for example [23, 24, 25].

In this paper all the walks, paths and cycles will be directed and we consider that each digraph has a (fixed) colouring of the arcs.

A path (or a cycle) is called monochromatic if all of its arcs are coloured alike. A cycle is called a quasi-monochromatic cycle if, with at most one exception, all of its arcs are coloured alike. A set \(N \subseteq V(D)\) is said to be a kernel by monochromatic paths if it satisfies the following two conditions: (i) for every pair of different vertices \(u, v \in N\) there is no monochromatic path between them \((N\) is an independent set by monochromatic paths\) and; (ii) for every vertex \(x \in V(D) \setminus N\) there is a vertex \(y \in N\) such that there is a \(xy\)-monochromatic path \((N\) is an absorbing set by monochromatic paths\).

The definition of kernel by monochromatic paths was introduced by Galeana-Sánchez [9], even though the research on kernels by monochromatic paths goes back to the classical paper of Sands et al. [27], kernel by monochromatic paths clearly, is a generalization of the concept of kernel. The closure of \(D\), denoted by \(\mathcal{C}(D)\) is the \(m\)-coloured multidigraph defined as follows: \(V(\mathcal{C}(D)) = V(D)\), \(A(\mathcal{C}(D)) = A(D) \cup \{(u, v) \mid \text{with colour } i \text{ there is a } uv\text{-path coloured } i \text{ contained}\)
in $D$}. Notice that for any digraph $D$, $C'(C'(D)) \cong C'(D)$ and $D$ has a kernel by monochromatic paths if and only if $C'(D)$ has a kernel.

In [27] Sands et al. have proven that any 2-coloured digraph $D$ has a kernel by monochromatic paths; in particular they proved that any 2-coloured tournament $T$ has a kernel by monochromatic paths. They also raised the following problem: Let $T$ be a 3-coloured tournament such that every cycle of length 3 is a quasi-monochromatic cycle; must $T$ have a kernel by monochromatic paths? (This question still remains open.)

In [28] Shen Minggang proved that if $T$ is a $m$-coloured tournament such that every triangle (that is, a transitive tournament of order 3 or a cycle of length 3) is a quasimonochromatic subdigraph of $T$, then $T$ has a kernel by monochromatic paths. He also proved that this result is the best possible for $m \geq 5$. In [16] H. Galeana-Sánchez and R. Rojas Monroy proved that the result of Shen Minggang is the best possible for $m \geq 4$.

In [13] H. Galeana-Sánchez, R. Rojas-Monroy and G. Gaytán-Góme proved that if $D$ is a finite $m$-coloured digraph that admits a partition $\{C_1, C_2\}$ of the set of colours of $D$ such that for each $i \in \{1, 2\}$ every cycle in the subdigraph $D[C_i]$ spanned by the arcs with colours in $C_i$ is monochromatic, $C'(D)$ does not contain neither rainbow triangles (all of its arcs have different colours) nor rainbow $P_3$ (path of length 3) involving colours of both $C_1$ and $C_2$; then $D$ has a kernel by monochromatic paths.

The known sufficient conditions for the existence of a kernel by monochromatic paths ($k.m.p.$) in $m$-coloured ($m \geq 3$) tournaments or nearly tournaments (such as digraphs obtained from a tournament by the deletion of a single arc, quasi-transitive digraphs, $k$-partite tournaments) ask for the monochromaticity or quasi-monochromaticity of small subdigraphs such as directed cycles or transitive tournaments of order 3. Other interesting results about the existence of $k.m.p.$ in digraphs can be found in [9, 10, 11, 12, 17, 22, 29, 30].

If $W = (z_0, z_1, \ldots, z_n)$ is a walk, we say that the length of $W$ is $n$ and we will denote it by $\ell(W)$. If $P$ is a path and $z_i, z_j \in V(P)$ with $i \leq j$ we denote by $(z_i, P, z_j)$ the $z_iz_j$-path contained in $P$, and $\ell(z_i, P, z_j)$ will denote its length.

We will need the following basic elementary results.

**Lemma 1.** Let $D$ be a digraph, $u, v \in V(D)$. Then every $uv$-monochromatic walk in $D$ contains a $uv$-monochromatic path.

**Lemma 2.** Every closed walk in a digraph $D$ contains a cycle.

And the following theorem.

**Theorem 3** [3]. If $D$ is a digraph such that every cycle of $D$ has at least one symmetrical arc, then $D$ is a kernel-perfect digraph.
2. Main Results

Definition. Let \( D \) be a \( m \)-coloured digraph, a \( \gamma \)-cycle in \( D \) is a sequence of vertices \( \gamma = (u_0, u_1, \ldots, u_n) \) such that

1. \( u_i \neq u_j \) for each \( i \neq j \),
2. for each \( i \in \{0, 1, \ldots, n\} \) there is a \( u_iu_{i+1} \)-monochromatic path in \( D \) (the indices are taken \( \mod n+1 \)), and
3. for each \( i \in \{0, 1, \ldots, n\} \) there is no \( u_{i+1}u_i \)-monochromatic path.

We will say that the length of \( \gamma \) is \( \ell(\gamma) = n \).

A digraph \( D \) is called \emph{transitive by monochromatic paths} if the existence of an \( xy \)-monochromatic path and a \( yz \)-monochromatic path in \( D \) imply that there is an \( xz \)-monochromatic path in \( D \).

The following lemmas will be useful in the proof of our main result.

Lemma 4. Let \( D \) be a \( m \)-coloured and transitive by monochromatic paths digraph, then \( D \) has no \( \gamma \)-cycles.

Proof. Let \( C = (u_0, u_1, \ldots, u_{n-1}, u_0) \) be a sequence of vertices such that \( u_i \neq u_j \) for each \( i \neq j \), and for every \( i \in \{0, 1, \ldots, n-1\} \) there is a \( u_iu_{i+1} \)-monochromatic path in \( D \) (the indices of the vertices will be taken \( \mod n \)). We can prove, by induction and from transitivity by monochromatic paths that there exists a \( u_0u_k \)-monochromatic path in \( D \) for each \( k \in \{2, \ldots, n-1\} \).

Then, there is a \( u_0u_{n-1} \)-monochromatic path in \( D \). We conclude that \( D \) has no \( \gamma \)-cycles.

Lemma 5. Let \( D \) be a \( m \)-coloured digraph such that has no \( \gamma \)-cycles. Then there is no sequence of vertices \( (x_0, x_1, x_2, \ldots) \) such that for every \( i \) there is an \( x_ix_{i+1} \)-monochromatic path in \( D \) and there is no \( x_{i+1}x_i \)-monochromatic path in \( D \).

Proof. It follows immediately from the finiteness of \( D \).

Definition. Let \( D \) be an \( m \)-coloured digraph. A set \( S \subseteq V(D) \) is a \emph{semikernel by monochromatic paths} of \( D \) if the following conditions are fulfilled:

1. \( S \) is an independent set by monochromatic paths, and
2. for each \( z \in V(D) \setminus S \) such that there exists an \( Sz \)-monochromatic path, then there exists a \( zS \)-monochromatic path in \( D \).

Lemma 6. Let \( D \) be an \( m \)-coloured digraph such that has no \( \gamma \)-cycles. Then there exists \( x_0 \in V(D) \) such that \( \{x_0\} \) is a semikernel by monochromatic paths of \( D \).
\textbf{Proof.} If there exists no vertex that satisfies the affirmation of Lemma 6, it is straightforward to build a vertex sequence that contradicts Lemma 5. \hfill ■

From now on, \( D \) will denote a finite \( m \)-coloured digraph and \( \{ C_1, C_2 \} \) will be a partition of \( C \), the set of colours of \( D \). Also, \( D_i \) will be the spanning subdigraph of \( D \) such that \( A(D_i) = \{ a \in A(D) \mid \text{colour}(a) \in C_i \} \). If \( W = (u_0, \ldots , u_k = v_0, \ldots , v_m = w_0, \ldots , w_n = u_0) \) is a cycle, we say that \( W \) is a \textit{3-coloured \( (C_1, C_2) \) subdivision of} \( C_3 \) (cycle of length 3) if \( T_1 = (u_0, \ldots , u_k) \) is a monochromatic path of colour \( a \) and it is contained in \( D_1 \), \( T_2 = (v_0, \ldots , v_m) \) is a monochromatic path of colour \( b \) and it is contained in \( D_2 \), and \( T_3 = (w_0, \ldots , w_n) \) is a monochromatic path of colour \( c \) and it is contained in \( D_3 \) with \( a \neq b, b \neq c \), and \( a \neq c \). And, if \( P = (u_0, \ldots , u_k = v_0, \ldots , v_m = u_0, \ldots , w_n) \) is a directed path, we say that \( P \) is a \textit{3-coloured \( (C_1, C_2) \) subdivision of} \( P \) if \( T_1 = (u_0, \ldots , u_k) \) is a monochromatic path of colour \( a \) and it is contained in \( D_1 \), \( T_2 = (v_0, \ldots , v_m) \) is a monochromatic path of colour \( b \) and it is contained in \( D_2 \), and \( T_3 = (w_0, \ldots , w_n) \) is a monochromatic path of colour \( c \) and it is contained in \( D_3 \) with \( a \neq b, b \neq c \), and \( a \neq c \). In particular, we say that a cycle \((u_0, u_1, u_2, u_0)\) is a \textit{3-coloured \( (C_1, C_2) \) subdivision of} \( C_3 \) if \( a = \text{colour}((u_0, u_1)) \in C_1, b = \text{colour}((u_1, u_2)) \in C_1 \) and \( c = \text{colour}((u_2, u_0)) \in C_2 \) with \( a \neq b, b \neq c \), and \( a \neq c \). We say that a path \((u_0, u_1, u_2, u_3)\) is a \textit{3-coloured \( (C_1, C_2) \) subdivision of} \( P \) if \( a = \text{colour}((u_0, u_1)) \in C_1, b = \text{colour}((u_1, u_2)) \in C_1 \) and \( c = \text{colour}((u_2, u_0)) \in C_2 \) with \( a \neq b, b \neq c \), and \( a \neq c \). We say that a vertex \((u, v)\) in \( V(D) \) is a \textit{3-coloured \( (C_1, C_2) \) in-neighbourhood} if there exists \( w, x \) and \( z \) in \( V(D) \) such that \( \{ (w, v), (x, v), (z, v) \} \subseteq A(D) \) and \( a = \text{colour}((w, v)) \in C_1 \), \( b = \text{colour}((x, v)) \in C_1 \) and \( c = \text{colour}((z, v)) \in C_2 \) with \( a \neq b, b \neq c \), and \( a \neq c \).

**Definition.** Let \( S \subseteq V(D) \). We will say that \( S \) is a \textit{semikernel by monochromatic paths modulo} \( D_2 \) of \( D \) if \( S \) is independent by monochromatic paths and for every \( z \in V(D) \setminus S \), if there is an \( Sz \)-monochromatic path contained in \( D_1 \) then there is a \( zS \)-monochromatic path contained in \( D \).

**Lemma 7.** Suppose that \( D_1 \) has no \( \gamma \)-cycles. Then there exists \( x_0 \in V(D) \) such that \( \{ x_0 \} \) is a semikernel by monochromatic paths modulo \( D_2 \) of \( D \).

**Proof.** Since \( D_1 \) has no \( \gamma \)-cycles, then it follows from Lemma 6 that there exists \( x_0 \in V(D_1) \) such that \( \{ x_0 \} \) is a semikernel by monochromatic paths of \( D_1 \). From the definition of semikernel by monochromatic paths modulo \( D_2 \) of \( D \), we have that \( \{ x_0 \} \) is a semikernel by monochromatic paths modulo \( D_2 \) of \( D \). \hfill ■

Let \( \varsigma = \{ \emptyset \neq S \subseteq V(D) \mid S \text{ is a semikernel by monochromatic paths modulo } D_2 \text{ of } D \} \).

Whenever \( \varsigma \neq \emptyset \), we will denote by \( D_\varsigma \) the digraph defined as follows: \( V(D_\varsigma) = \varsigma \) (i.e., for every element of \( \varsigma \) we consider a vertex in \( D_\varsigma \)) and \( (S_1, S_2) \in A(D_\varsigma) \) if and only if for every \( s_1 \in S_1 \) there exists \( s_2 \in S_2 \) such that \( s_1 = s_2 \).
or there is an $s_1s_2$-monochromatic path contained in $D_2$ and there is no $s_2S_1$-monochromatic path contained in $D$.

**Lemma 8.** Suppose that:

1. $D_1$ has no $\gamma$-cycles, and
2. $D_2$ is transitive by monochromatic paths.

Then $D_\varsigma$ is an acyclic digraph.

**Proof.** First we will prove that $D_\varsigma$ is transitive and anti-symmetric.

Transitive. Suppose that $(S,T) \in A(D_\varsigma)$ and $(T,W) \in A(D_\varsigma)$, and let $s \in S$. If $s \notin W$, we may suppose $s \notin T$ as well from $(T,W) \in A(D_\varsigma)$, and so there is a monochromatic path contained in $D_2$ from $s$ to some $t \in T$. If $t \in W$, we are done. Otherwise there is a monochromatic path contained in $D_2$ from $t$ to some $w \in W$. Then since $D_2$ is transitive by monochromatic paths there is a monochromatic path from $s$ to $w$. Then $(S,W) \in A(D_\varsigma)$.

Anti-symmetric. Suppose that $(S,T) \in A(D_\varsigma)$ and $(T,S) \in A(D_\varsigma)$ we will prove that $S = T$. Proceeding by contradiction, suppose, without loss of generality, that $s \in S \setminus T$. Then there is a monochromatic path contained in $D_2$ from $s$ to some $t \in T$ and there is no $tS$-monochromatic path contained in $D$. Since $(T,S) \in A(D_\varsigma)$ then $t$ must belong to $S$, a contradiction because $s \in S$, $t \in S$ and $S$ is independent by monochromatic paths. Then $S = T$.

Now assume, for a contradiction, that $D_\varsigma$ has a cycle, say $\mathcal{C} = (S_0, S_1, \ldots, S_{n-1}, S_0)$, with $n \geq 2$. Since $\mathcal{C}$ is a cycle, we have that $S_i \neq S_j$ if $i \neq j$. We can prove, by induction and from transitivity that $(S_{i+1}, S_i) \in A(D_\varsigma)$ for each $i \in \{0, 1, \ldots, n - 1\}$ (the indices of the vertices will be taken mod $n$). Since $D_\varsigma$ is anti-symmetric we have $S_i = S_j$, a contradiction. We conclude that $D_\varsigma$ is an acyclic digraph. \vspace{10pt}

**Lemma 9.** Suppose that $\alpha_1$ is a $uw$-monochromatic path in $D_1$, $\alpha_2$ is a $zw$-monochromatic path in $D_1$ and $\alpha_3$ is a $wx$-monochromatic path in $D_2$, such that $\text{colour}(\alpha_1) \neq \text{colour}(\alpha_2)$, $\text{colour}(\alpha_1) \neq \text{colour}(\alpha_3)$ and $\text{colour}(\alpha_2) \neq \text{colour}(\alpha_3)$. Additionally, assume that $D$ has no $uw$-monochromatic path, no $zx$-monochromatic path, and no $zu$-monochromatic path. Then each one of the two following conditions imply that there is a $uw$-path which is a $3$-coloured $(C_1, C_1, C_2)$ subdivision of $P_3$ or there is a $3$-coloured $(C_1, C_1, C_2)$ subdivision of $C_3$:

(a) Each cycle of $D$ contained in $D_1$ is monochromatic and $D_2$ is transitive by monochromatic paths.

(b) $D$ has no vertex with $3$-coloured $(C_1, C_1, C_2)$ in-neighbourhood.

**Proof.** From the hypothesis, we have immediately the following assertions:

1. $u \notin V(\alpha_2)$. 

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\((2)\) \(z \notin V(\alpha_3)\).
\((3)\) \(w \notin V(\alpha_1)\).
\((4)\) \(x \notin V(\alpha_2)\).

**Case I.** \(V(\alpha_1) \cap V(\alpha_2) = \{z\} \).

**Subcase I.1.** \(V(\alpha_2) \cap V(\alpha_3) = \{w\} \).

**Subcase I.1.1.** \(V(\alpha_1) \cap V(\alpha_3) = \emptyset \). In this case, we have that \(\alpha_1 \cup \alpha_2 \cup \alpha_3\) is a \(ux\)-path which is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{P}_3\).

**Subcase I.1.2.** \(V(\alpha_1) \cap V(\alpha_3) \neq \emptyset\). Let \(y\) be the last vertex of \(\alpha_1\) which is in \(\alpha_3\). We have that \(y \neq z\) and \(w \neq y\) (from assertions 2 and 3). Then \((y, \alpha_1, z) \cup \alpha_2 \cup (w, \alpha_3, y)\) is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{C}_3\).

**Subcase I.2.** \(V(\alpha_2) \cap V(\alpha_3) \neq \emptyset\). Let \(y\) be the first vertex of \(\alpha_2\) that is in \(\alpha_3\). We have that \(y \neq z\) and \(y \neq x\) (from assertions 2 and 4). Then \(\alpha_3 \cup (z, \alpha_2) \cup (y, \alpha_3, x)\) is a \(ux\)-path which is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{P}_3\).

**Subcase I.2.1.** \(V(\alpha_1) \cap V(\alpha_3) = \emptyset\). Let \(y\) be the first vertex of \(\alpha_3\) that is in \(\alpha_1\) or in \(\alpha_2\) and let \(e\) be the last vertex of \(\alpha_3\) which is in \(\alpha_1\) or in \(\alpha_2\). If \(y \in V(\alpha_1)\), then we have that \(y \neq z\) (from assertion 2) and we may suppose that \(y \neq w\) (from assertion 3). Then \((y, \alpha_1, z) \cup \alpha_2 \cup (w, \alpha_3, y)\) is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{C}_3\). Suppose that \(y \notin V(\alpha_1)\), then \(y \in V(\alpha_2)\). If \(e \in V(\alpha_1)\), then \(y \neq e\). Let \(a\) be the last vertex of \(\alpha_3\) which is in \(\alpha_2\) and let \(b\) be the first vertex of \((a, \alpha_3, x)\) that is in \(\alpha_1\). We have that \(b \neq z\) and \(a \neq z\) (from assertion 2), and \(a \neq x\) (from assertion 4), \(a \neq u\) (from assertion 1) and \(b \neq w\) (from assertion 3). Also, \(a \neq b\), otherwise \((a, \alpha_1, z) \cup (z, \alpha_2, a)\) contains a non-monochromatic cycle contained in \(D_1\) and \(a\) is a vertex with 3-coloured \((C_1, C_1, C_2)\) in-neighbourhood, a contradiction. Then \((b, \alpha_1, z) \cup (z, \alpha_2, a) \cup (a, \alpha_3, b)\) is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{C}_3\). Now, assume that \(e \in V(\alpha_2)\). We have that \(z \neq e\) and \(e \neq x\) (from assertions 2 and 4). Then \(\alpha_1 \cup (z, \alpha_2, e) \cup (e, \alpha_3, x)\) is a \(ux\)-path which is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{P}_3\).

**Case II.** \((V(\alpha_1) \cap V(\alpha_2)) \setminus \{z\} \neq \emptyset\). Suppose that \(D\) satisfies (a), \((V(\alpha_1) \cap V(\alpha_2)) \setminus \{z\} \neq \emptyset\) implies that there is a non-monochromatic cycle contained in \(\alpha_1 \cup \alpha_2 \subseteq D_1\), a contradiction. Therefore, \(D\) satisfies (b).

**Subcase II.1.** \(V(\alpha_2) \cap V(\alpha_3) = \{w\}\).

**Subcase II.1.1.** \(V(\alpha_1) \cap V(\alpha_3) = \emptyset\). Let \(y\) be the first vertex of \(\alpha_1\) that is in \(\alpha_2\). We have that \(y \neq u\) and \(y \neq w\) (from assertions 1 and 3). Then \((u, \alpha_1, y) \cup (y, \alpha_2, w) \cup \alpha_3\) is a 3-coloured \(ux\)-path which is a \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{P}_3\).
Subcase II.1.2. \( V(\alpha_1) \cap V(\alpha_3) \neq \emptyset \). Let \( y \) be the first vertex of \( \alpha_1 \) that is in \( \alpha_2 \) or \( \alpha_3 \) and let \( e \) be the last vertex of \( \alpha_1 \) that is in \( \alpha_2 \) or \( \alpha_3 \). If \( y \in V(\alpha_2) \) then we have that \( u \neq y \) and \( y \neq w \) (from assertions 1 and 3). Then \((u, \alpha_1, y) \cup (y, \alpha_2, w) \cup \alpha_3 \) is a \( uw \)-path which is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \( \overline{P}_3 \). Suppose that \( y \in V(\alpha_3) \). If \( e \in V(\alpha_2) \), let \( a \) be the last vertex of \( \alpha_1 \) that is in \( \alpha_3 \) and let \( b \) be the first vertex of \((a, \alpha_1, z)\) that is in \( \alpha_2 \). We have that \( b \neq w \) and \( a \neq w \) (from assertion 3), \( a \neq b \) (because \( V(\alpha_2) \cap V(\alpha_3) = \{w\} \)), and \( a \neq z \) (from assertion 2). Then \((a, \alpha_1, b) \cup (b, \alpha_2, w) \cup (w, \alpha_3, a)\) is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \( \overrightarrow{C}_3 \). So, assume that \( e \in V(\alpha_3) \). We have that \( e \neq z \) and \( e \neq w \) (from assertions 2 and 3). Then \((e, \alpha_1, z) \cup \alpha_2 \cup (w, \alpha_3, e)\) is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \( \overrightarrow{C}_3 \).

Subcase II.2. \( (V(\alpha_2) \cap V(\alpha_3)) \setminus \{w\} \neq \emptyset \).

Subcase II.2.1. \( V(\alpha_1) \cap V(\alpha_3) = \emptyset \). Let \( y \) be the first vertex of \( \alpha_2 \) that is in \( \alpha_1 \) or \( \alpha_3 \) and let \( e \) be the last vertex of \( \alpha_2 \) that is in \( \alpha_1 \) or \( \alpha_3 \). If \( y \in V(\alpha_3) \) then we have that \( y \neq z \) and \( y \neq x \) (from assertions 2 and 4). Then \( \alpha_1 \cup (z, \alpha_2, y) \cup (y, \alpha_3, x) \) is a \( uw \)-path which is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \( \overline{P}_3 \). Suppose that \( y \in V(\alpha_1) \). If \( e \in V(\alpha_1) \) then \( u \neq e \) and \( e \neq w \) (from assertions 1 and 3). Then, \((u, \alpha_1, e) \cup (e, \alpha_2, w) \cup \alpha_3 \) is a \( uw \)-path which is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \( \overline{P}_3 \). If \( e \in V(\alpha_2) \), then let \( a \) be the last vertex of \( \alpha_2 \) that is in \( \alpha_1 \) and let \( b \) be the first vertex of \((a, \alpha_2, w)\) that is in \( \alpha_3 \). We have that \( u \neq a \) and \( b \neq x \) (from assertions 1 and 4), and \( a \neq b \) \((V(\alpha_1) \cap V(\alpha_3) = \emptyset)\). Then \((u, \alpha_1, a) \cup (a, \alpha_2, b) \cup (b, \alpha_3, x)\) is a \( uw \)-path which is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \( \overline{P}_3 \).

Subcase II.2.2. \( V(\alpha_1) \cap V(\alpha_3) \neq \emptyset \). Let \( a \) be the first vertex of \( \alpha_1 \) that is in \( \alpha_2 \) and let \( b \) be the first vertex of \((a, \alpha_2, w)\) which is in \( \alpha_3 \). Then, we have that \( u \neq a \) and \( b \neq x \) (from assertions 1 and 4), \( a \neq w \) (from assertion 3), and \( b \neq z \) (from assertion 2). Also, \( a \neq b \), otherwise \((a, \alpha_1, z) \cup (z, \alpha_2, b)\) contains a non-monochromatic cycle in \( D_1 \) and \( a \) is a vertex with 3-coloured \((C_1, C_1, C_2)\) in-neighbourhood, a contradiction. Suppose that \( V((b, \alpha_3, x)) \cap V(\{u, \alpha_1, a\}) = \emptyset \). Then, \((u, \alpha_1, a) \cup (a, \alpha_2, b) \cup (b, \alpha_3, x)\) is a \( uw \)-path which is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \( \overline{P}_3 \). If \( V((b, \alpha_3, x)) \cap (u, \alpha_1, a) \neq \emptyset \), let \( c \) be the first vertex of \((b, \alpha_3, x)\) that is in \((u, \alpha_1, a)\). Since \( a \neq b \) then the definitions of \( a \) and \( b \) imply that \( c \neq a \) and \( c \neq b \). Then \((c, \alpha_1, a) \cup (a, \alpha_2, b) \cup (b, \alpha_3, c)\) is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \( \overrightarrow{C}_3 \). □

Definition. We say that the digraph \( D \) satisfies the property \( A \) if:

1. \( D_1 \) has no \( \gamma \)-cycles, and
2. \( \mathcal{C}(D) \) possesses the following two conditions:
   i. every 3-coloured \((C_1, C_1, C_2) - \overrightarrow{C}_3 \) has at least two symmetrical arcs,
(ii) if \((u, z, w, x)\) is a 3-coloured \((C_1, C_1, C_2) - \overrightarrow{P}_3\) then \((u, x) \in A(\mathcal{E}(D))\).

**Definition.** We say that the digraph \(D\) satisfies the **property B** if:
1. Every cycle contained in \(D_1\) is monochromatic,
2. \(D\) contains no 3-coloured \((C_1, C_1, C_2)\) subdivisions of \(\overrightarrow{C}_3\), and
3. If \((u, z, w, x)\) is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{P}_3\) then there is a monochromatic path between \(u\) and \(x\) in \(D\).

**Definition.** We say that the digraph \(D\) satisfies the **property C** if:
1. \(D_1\) has no \(\gamma\)-cycles,
2. \(D\) has no vertices with 3-coloured \((C_1, C_1, C_2)\) in-neighbourhood,
3. \(D\) contains no 3-coloured \((C_1, C_1, C_2)\) subdivisions of \(\overrightarrow{C}_3\), and
4. If \((u, z, w, x)\) is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{P}_3\) then there is a monochromatic path between \(u\) and \(x\) in \(D\).

**Theorem 10.** Suppose that \(D_2\) is transitive by monochromatic paths. If \(D\) satisfies one of the properties \(A, B\) or \(C\), then \(D\) has a k.m.p.

**Proof.** Consider the digraph \(D\). Note that if every cycle in a digraph is monochromatic then such digraph contains no \(\gamma\)-cycles. So, in any case \(D_1\) has no \(\gamma\)-cycles. Thus, Lemma 8 implies that \(D_1\) is acyclic. Then \(D_1\) contains at least one vertex with zero outdegree. Let \(S \in V(D_1)\) be such that \(\delta^+_D(S) = 0\). We will prove, by contradiction, that \(S\) is a k.m.p. of \(D\).

Since \(S \in V(D_1)\), then \(S\) is independent by monochromatic paths. If \(S\) is not a k.m.p., then \(S\) is not absorbent by monochromatic paths. Let \(X = \{z \in V(D) \mid \text{there is no } zS\text{-monochromatic path in } D\}\). From our assumption we obtain \(X \neq \emptyset\). Given that \(D[X]\) is an induced subdigraph of \(D\), we have that \(D[X]\) satisfies the hypothesis of Theorem 10 and the subdigraph of \(D_1\) contained in \(D[X]\) satisfies the hypothesis of Lemma 7. It follows that there exists \(x_0 \in X\) such that \(\{x_0\}\) is a semikernel by monochromatic paths modulo \(D_2\) of \(D[X]\).

Let \(T = \{z \in S \mid \text{there is no } zx_0\text{-monochromatic path in } D_2\}\). From the definition of \(T\), we have that for each \(z \in S \setminus T\) there is a \(zx_0\)-monochromatic path contained in \(D_2\).

Note that each monochromatic path of \(D\) is contained either in \(D_1\) or in \(D_2\).

**Claim 1.** \(T \cup \{x_0\}\) is independent by monochromatic paths.

**Proof.** \(T\) is independent by monochromatic paths because \(T \subseteq S\) and \(S \in \varsigma\).

There is no \(Tx_0\)-monochromatic path contained in \(D\). Otherwise, from the definition of \(T\), such path must be contained in \(D_1\). Since \(T \subseteq S \in \varsigma\) then there is a \(x_0S\)-monochromatic path, but this contradicts the definition of \(X\).

There is no \(x_0T\)-monochromatic path. It follows from the definition of \(X\).

We conclude that \(T \cup \{x_0\}\) is independent by monochromatic paths. \(\square\)
Claim 2. If there is a \((T \cup \{x_0\})\)-monochromatic path contained in \(D_1\) then there is a \(z(T \cup \{x_0\})\)-monochromatic path.

Proof. We have two cases.

Case 1. There is a \(Tz\)-monochromatic path contained in \(D_1\). Since \(T \subseteq S\) and \(S \in D_\zeta\), it follows that there is a \(zS\)-monochromatic path contained in \(D\). We may suppose that such path is a \(z(S \setminus T)\)-monochromatic path. Let \(\alpha_1\) be a \(uz\)-monochromatic path contained in \(D_1\) with \(u \in T\) and let \(\alpha_2\) be a \(zw\)-monochromatic path contained in \(D\) with \(w \in S \setminus T\). Since \(w \in S \setminus T\), the definition of \(T\) implies that there is a \(wx_0\)-monochromatic path contained in \(D_2\), say \(\alpha_3\). First, suppose that \(\alpha_2 \subseteq D_2\), since \(D_2\) is transitive by monochromatic paths then there is a \(zx_0\)-monochromatic path contained in \(D_2\). So, we may suppose that \(\alpha_2 \nsubseteq D_1\). If \(\text{colour}(\alpha_1) = \text{colour}(\alpha_2)\), then \(\alpha_1 \cup \alpha_2\) contains a \(uw\)-monochromatic path, a contradiction as \(\{u, w\} \subseteq S\) and \(S \in \zeta\). Hence, \(\text{colour}(\alpha_1) \neq \text{colour}(\alpha_2)\). Moreover, \(\text{colour}(\alpha_1) \neq \text{colour}(\alpha_3)\) (\(\alpha_1 \subseteq D_1\) and \(\alpha_3 \subseteq D_2\)) and \(\text{colour}(\alpha_2) \neq \text{colour}(\alpha_3)\) (\(\alpha_2 \subseteq D_1\) and \(\alpha_3 \subseteq D_2\)).

If \(D\) satisfy the property \(A\), then \((u, z, w, x_0)\) is a path in \(\mathcal{E}(D)\) which is a 3-coloured \((C_1, C_1, C_2) - \overrightarrow{P_3}\). By hypothesis \((u, x_0) \in \mathcal{A}(\mathcal{E}(D))\), then, we have a \(ux_0\)-monochromatic path in \(D\); a clear contradiction because \(u \in T\) and \(T \cup \{x_0\}\) is independent by monochromatic paths. So, assume that \(D\) satisfies one of the properties \(B\) or \(C\).

Now, note that: There is no \(uw\)-monochromatic path. It follows from \(\{u, w\} \subseteq S\) and \(S\) is independent by monochromatic paths.

We may suppose that there is no \(zx_0\)-monochromatic path and there is no \(zw\)-monochromatic path, otherwise there is a \(z(T \cup \{x_0\})\)-monochromatic path.

Then \(D, \alpha_1, \alpha_2\) and \(\alpha_3\) satisfies the hypothesis of Lemma 9. In any case:

- there is a \(ux_0\)-path which is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{P_3}\) or
- there is a 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{C_3}\).

In the first case, we have that there is a monochromatic path between \(u\) and \(x_0\) in \(D\). But, this contradicts the fact that \(T \cup \{x_0\}\) is independent by monochromatic paths. The second case is not possible since \(D\) contains no 3-coloured \((C_1, C_1, C_2)\) subdivision of \(\overrightarrow{C_3}\).

Case 2. There is an \(x_0z\)-monochromatic path contained in \(D_1\). Let \(\alpha_1\) be an \(x_0z\)-monochromatic path contained in \(D_1\). From the choice of \(x_0\) we may suppose that \(z \notin X\). Then, the definition of \(X\) implies that there is a \(zS\)-monochromatic path contained in \(D\), say \(\alpha_2\). Suppose that \(\alpha_2\) ends in \(w\). If \(w \in T\) then \(\alpha_2\) is a \(z(T \cup \{x_0\})\)-monochromatic path in \(D\). Then, suppose that \(w \in S \setminus T\). From the definition of \(T\) it follows that there is a \(wx_0\)-monochromatic path contained in \(D_2\), call \(\alpha_3\) such path. Assume that \(\alpha_2 \subseteq D_2\), since \(D_2\) is transitive
by monochromatic paths then there is a $xz_0$-monochromatic path contained in $D_2$. So, we may suppose that $\alpha_2 \subseteq D_1$. If $\text{colour}(\alpha_1) = \text{colour}(\alpha_2)$ then $\alpha_1 \cup \alpha_2$ contains an $x_0w$-monochromatic path, a contradiction with the definition of $X$. Hence, $\text{colour}(\alpha_1) \neq \text{colour}(\alpha_2)$. Furthermore, $\text{colour}(\alpha_1) \neq \text{colour}(\alpha_3)$ ($\alpha_1 \subseteq D_1$ and $\alpha_3 \subseteq D_2$) and $\text{colour}(\alpha_2) \neq \text{colour}(\alpha_3)$ ($\alpha_2 \subseteq D_1$ and $\alpha_3 \subseteq D_2$).

Suppose that $D$ satisfies the property A, then $\mathcal{C}(D)$ contains a 3-coloured $(C_1, C_1, C_2) - \overrightarrow{C_3}$ (to be explicit: $(x_0, z, w, x_0)$), then this $\overrightarrow{C_3}$ has at least two symmetrical arcs. Then $(z, x_0) \in A(\mathcal{C}(D))$ or $(x_0, w) \in A(\mathcal{C}(D))$. If $(z, x_0) \in A(\mathcal{C}(D))$, then we have a $xz_0$-monochromatic path in $D$ and Claim 2 is proved. If $(x_0, w) \in A(\mathcal{C}(D))$, then we have a $x_0w$-monochromatic path in $D$, contradicting the definition of $X$.

Now, suppose that $D$ satisfies one of the properties B or C. Let $u = x_0$, note that: there is no $uw$-monochromatic path. It follows from the definition of $X$.

We may suppose that there is no $zu$-monochromatic path. Then $D, \alpha_1, \alpha_2$ and $\alpha_3$ satisfies the hypothesis of Lemma 9. In any case: there is a $ux_0$-path which is a 3-coloured $(C_1, C_1, C_2)$ subdivision of $\overrightarrow{P_3}$ or

There is a 3-coloured $(C_1, C_1, C_2)$ subdivision of $\overrightarrow{C_3}$.

The first case is not possible as $u = x_0$. The second case is not possible since $D$ contains no 3-coloured $(C_1, C_1, C_2)$ subdivision of $\overrightarrow{C_3}$.

It follows from Claim 1 and Claim 2 that $(T \cup \{x_0\}) \in \varsigma$, so, $(T \cup \{x_0\}) \in V(D_\varsigma)$.

Now, since $T \subseteq S$, $x_0 \in X$ and for each $s \in S$ such that $s \notin T$ there is an $sx_0$-monochromatic path contained in $D_2$ and there is no $x_0s$-monochromatic path contained in $D$ then $(S, T \cup \{x_0\}) \in A(D_\varsigma)$. We obtain a contradiction with the assumption $\delta^+_D(S) = 0$.

We conclude that $S$ is a k.m.p. of $D$.\[\square\]

**Remark 11.** Notice that Theorem 10 generalizes the theorem of Sands, Sauer and Woodrow since:

1. A 2-coloured digraph can be divided in two monochromatic spanning subdigraphs $D_1 = D[\{a \in A(D) \mid \text{colour}(a) = \text{colour 1}\}]$ and $D_2 = D[\{a \in A(D) \mid \text{colour}(a) = \text{colour 2}\}]$.
2. Every directed cycle in $D_1$ is monochromatic since $D_1$ is monochromatic.
3. $D_1$ has no $\gamma$-cycles since $D_1$ is monochromatic.
4. $D_2$ is transitive by monochromatic paths since $D_2$ is monochromatic.

Now, since only two colours are used on $D$, then we have the following assertions.

5. $\mathcal{C}(D)$ satisfies the following two conditions:

   - (i) all 3-coloured $\overrightarrow{C_3} - (C_1, C_1, C_2)$ has at least two symmetrical arcs,
   - (ii) if $(u, v, w, x)$ is a 3-coloured $\overrightarrow{P_3} - (C_1, C_1, C_2)$ then $(u, x) \in A(\mathcal{C}(D))$.

6. $D$ has no vertices with 3-coloured $(C_1, C_1, C_2)$ in-neighbourhood.
(7) $D$ contains no 3-coloured $(C_1, C_1, C_2)$ subdivisions of $\overrightarrow{C_3}$.
(8) If $(u, v, w, x)$ is a 3-coloured $(C_1, C_1, C_2)$ subdivision of $\overrightarrow{P_3}$ then there is a monochromatic path in $D$ between $u$ and $x$.

Therefore, every 2-coloured digraph $D$ fulfils the hypotheses of our main theorem, furthermore it satisfies the three properties A, B and C. We conclude that theorem generalizes the theorem of Sands, Sauer and Woodrow.

With Theorem 10 we can generate new theorems, for example, let $D_1$ be a tournament that satisfies the hypothesis of Shen Minggang’s theorem, then it is possible to prove that $D_1$ has no $\gamma$-cycles. Then we obtain the following new theorem.

**Theorem 12.** Let $D$ be an $m$-coloured digraph such that:

1. $D_1$ is a tournament such that every triangle is a quasi-monochromatic subdigraph of $D_1$.
2. $D_2$ is transitive by monochromatic paths.
3. $\mathcal{C}(D)$ has the following two conditions:
   i. every 3-coloured $(C_1, C_1, C_2) - \overrightarrow{C_3}$ has at least two symmetrical arcs,
   ii. if $(u, v, w, x)$ is a 3-coloured $(C_1, C_1, C_2) - \overrightarrow{P_3}$ then $(u, x) \in A(\mathcal{C}(D))$.

Then $D$ has a k.m.p.

Similarly, it is possible to generate new theorems if $D_1$ is one of the following digraphs:

- (H. Galeana-Sánchez and J.J. García-Ruvalcaba, [11]) An $m$-coloured digraph resulting from the deletion of the single arc $(x, y)$ from some $m$-coloured tournament such that every triangle is quasi-monochromatic.
- (H. Galeana-Sánchez, R. Rojas Monroy, [17]) An $m$-coloured bipartite tournament such that every directed cycle of length 4 is monochromatic.
- (H. Galeana-Sánchez and R. Rojas Monroy, [19]) An $m$-coloured $k$-partite tournament with each cycle of length 3 and each cycle of length 4 monochromatic.
- (Gena Hahn, Pierre Ille and Robert E. Woodrow, [22]) A finite $k$-coloured tournament satisfying:
  - every tournament on 3 vertices is quasi-monochromatic, and
  - for $s \geq 4$, each cycle of length $s$ is quasi-monochromatic and no cycle of length less than $s$ has at least three colours on its arcs.
Other conditions which imply that an \( m \)-coloured digraph has no \( \gamma \)-cycles can be found in [4, 5, 18, 20, 21].

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\[ \gamma \text{-cycles and Transitivity by Monochromatic Paths in Digraphs} \]


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