THE WIENER NUMBER OF POWERS OF THE MYCIELSKIAN

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Abstract

The Wiener number of a graph $G$ is defined as $\frac{1}{2} \sum_{u,v \in V(G)} d(u, v)$, $d$ the distance function on $G$. The Wiener number has important applications in chemistry. We determine a formula for the Wiener number of an important graph family, namely, the Mycielskians $\mu(G)$ of graphs $G$. Using this, we show that for $k \geq 1$, $W(\mu(S_n^k)) \leq W(\mu(T_n^k)) \leq W(\mu(P_n^k))$, where $S_n$, $T_n$ and $P_n$ denote a star, a general tree and a path on $n$ vertices respectively. We also obtain Nordhaus-Gaddum type inequality for the Wiener number of $\mu(G^k)$.

Keywords: Wiener number, Mycielskian, powers of a graph.

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1. Introduction

Let $G$ be a simple connected undirected graph with vertex set $V(G)$ and edge set $E(G)$. Then $G$ is of order $|V(G)|$ and size $|E(G)|$. Given two distinct vertices $u, v$ of $G$, let $d(u, v)$ denote the distance between $u$ and $v$ (= number of edges in a shortest path between $u$ and $v$ in $G$). The Wiener number (also called Wiener index) $W(G)$ of the graph $G$ is defined by

$$W(G) = \frac{1}{2} \sum_{a, b \in V(G)} d(a, b) = \sum_{i=1}^{D} ip(i, G),$$
where $p(i, G)$ denotes the number of pairs of vertices which are at distance $i$ in $G$, and $D$ is the diameter of $G$. The Wiener number is one of the oldest molecular-graph based structure-descriptors, first proposed by the American chemist Harold Wiener [13] as an aid to determine the boiling point of paraffins. Some of the recent articles in this topic are ([1, 2, 3, 4, 5, 7] and [14]).

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [11] developed an interesting graph transformation as follows. For a graph $G = (V, E)$, the Mycielskian of $G$ is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and is disjoint from $V$, and edge set $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex $x'$ is called the twin of the vertex $x$ (and $x$ the twin of $x'$) and the vertex $u$ is the root of $\mu(G)$. In recent times, there has been an increasing interest in the study of Mycielskians, especially, in the study of their circular chromatic numbers (see, for instance, [9, 6, 8] and [10]).

Let $H$ be a spanning connected subgraph of a (connected) graph $G$. Then for any pair of vertices $u, v$ of $G$, $d_G(u, v) \leq d_H(u, v)$. The $k$-th power of a graph $G$, denoted by $G^k$, is the graph with the same vertex set as $G$ and in which two vertices are adjacent if and only if their distance in $G$ is at most $k$. Clearly, $G^1 = G$.

The complement $\overline{G}$ of a graph $G$ is the graph with the same vertex set as $G$ and in which two vertices $u, v$ are adjacent if and only if $u, v$ are non-adjacent in $G$. In 1956, Nordhaus and Gaddum [12] gave bounds for the sum of the chromatic number $\chi(G)$ of a graph $G$ and its complement $\overline{G}$ as follows,

**Theorem 1.1.** For a graph $G$ of order $n$, $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.

Zhang and Wu [15] presented the corresponding Nordhaus-Gaddum (in short NG) type inequality for the Wiener number as:

**Theorem 1.2.** Let $G$ be a connected graph of order $n \geq 5$ with connected complement $\overline{G}$. Then $3\left(\frac{n}{2}\right) \leq W(G) + W(\overline{G}) \leq \frac{n^3 + 3n^2 + 2n - 6}{6}$.

The bounds in Theorem 1.2 are sharp.
2. Wiener Number of the Mycielskian of a Graph

We start this section by obtaining a formula for the Wiener number of the Mycielskian of a graph.

**Theorem 2.1.** The Wiener number of the Mycielskian of a connected graph $G$ of order $n$ and size $m$ is given by $W(\mu(G)) = 6n^2 - n - 7m - 4p(2, G) - p(3, G)$.

**Proof.** By definition,

$$W(\mu(G)) = \frac{1}{2} \sum_{a, b \in V(\mu(G))} d(a, b).$$

Hence

$$W(\mu(G)) = \sum_{a, b \in V} d(a, b') + \sum_{a \in V} d(a, b) + \frac{1}{2} \sum_{a', b' \in V'} d(a', b')$$

$$+ \frac{1}{2} \sum_{a, b \in V} d(a, b) + \sum_{a \in V} d(a, b')$$

$$= \sum_1 + \sum_2 + \sum_3 + \sum_4 + \sum_5 \text{ (say)}.$$ 

One can observe that, $\sum_1 = n$, $\sum_2 = 2n$, $\sum_3 = 2\binom{n}{2}$. As distance between any pair of vertices in $V$ is at most 4 in $\mu(G)$, $\sum_4 = \sum_{i=1}^3 ip(i, G) + 4\left[\binom{n}{2} - \sum_{i=1}^3 p(i, G)\right]$. Now the maximum distance from any vertex in $V$ to any vertex in $V'$ is 3. Note that if $ab \in E$, then $ab', ba' \in E(\mu(G))$, that is, each edge of $G$ will contribute two edges between $V$ and $V'$. Also for every $a \in V$, $d(a, a') = 2$, and for every $a, b \in V$ such that $d(a, b) = 2$, we have $d(a, b') = d(b, a') = 2$. Thus $\sum_5 = 2n + 2\sum_{i=1}^3 ip(i, G) + 3\left[n^2 - n - 2\sum_{i=1}^2 p(i, G)\right]$ and therefore, $W(\mu(G)) = 6n^2 - n - 7m - 4p(2, G) - p(3, G)$.

This formula comes in handy when finding the Wiener number of $\mu(G)$ for which $p(2, G)$ and $p(3, G)$ are known even if the diameter of $G$ is very large.

In [1], X. An et al. have shown that $W(S_n^k) \leq W(T_n^k) \leq W(P_n^k)$, $k \geq 1$ where $S_n$, $P_n$ and $T_n$ denotes a star, a path and a tree other than a star and a path on $n$ vertices. The formula mentioned in Theorem 2.1 helps us in proving that $W(\mu(S_n^k)) \leq W(\mu(T_n^k)) \leq W(\mu(P_n^k))$ for any $k \geq 1$. However, this cannot be deduced from X. An’s result mentioned above. In fact, there are graphs $G$ and $H$ with same order and size such that $W(G) > W(H)$ and $W(\mu(G)) < W(\mu(H))$. For example, let $G$ be $C_6$ with a pendant edge attached at a pair of opposite vertices and $H$ be $C_7$ with a
single pendant edge, then \( W(G) = 62 \) and \( W(H) = 61 \) while \( W(\mu(G)) = 273 \) and \( W(\mu(H)) = 275 \).

**Theorem 2.2.** \( W(\mu(S^k_n)) \leq W(\mu(T^k_n)) \leq W(\mu(P^k_n)), \ k \geq 1 \).

**Proof.** By virtue of Theorem 2.1, the result in Theorem 2.2 is equivalent to
\[ A = 7p(1, S^k_n) + 4p(2, S^k_n) + p(3, S^k_n) \geq B = 7p(1, T^k_n) + 4p(2, T^k_n) + p(3, T^k_n) \geq C = 7p(1, P^k_n) + 4p(2, P^k_n) + p(3, P^k_n). \]

We first prove that \( A \geq B \). If \( k \geq 2 \), then \( S^k_n = K_n \) which implies that \( p(1, S^k_n) = \binom{n}{2} \geq \sum_{i=1}^{\sqrt{\frac{n}{3}}} p(i, T^k_n) \) and this inequality implies \( A \geq B \) (as \( 7 > 4 > 1 \)). If \( k = 1 \), then \( diam(S_n) = 2 \) and \( D = diam(T_n) \geq 2 \). This gives, \( p(2, S_n) = \frac{n}{2} - 1 \), and therefore \( 7p(1, S_n) + 4p(2, S_n) = 7p(1, T_n) + 4p(2, T_n) + p(3, T_n) \). Once again, \( A \geq B \).

Next we prove that \( B \geq C \) by induction on \( n \). \( B \geq C \) is obvious for \( n \leq 4 \). Let \( T_n \) be a tree of order \( n \geq 5 \) and let \( T_n = u_1 \ldots u_{n-1} \) be a path of order \( n \). Let \( P = uu_1 \ldots uu_d \) be a longest path of \( T_n \) \((d < n - 1)\). \( u \) is then a pendant vertex of \( T_n \), and \( T_n - \{u\} \) is a tree of order \( n - 1 \). By induction hypothesis, \( B \geq C \) for \( T_n - \{u\} \) and \( P_n - \{v\} \). Let \( p(a, i, G) \) denote the number of vertices in \( G \) that are at distance \( i \) from \( a \). Clearly, \( p(i, T^k_n) = p(i, T^k_n - \{u\}) + p(u, i, T^k_n) \). So it is enough to prove that \( 7p(u, 1, T^k_n) + 4p(u, 2, T^k_n) + p(u, 3, T^k_n) \geq 7p(v, 1, P^k_n) + 4p(v, 2, P^k_n) + p(v, 3, P^k_n) \).

We know that \( p(v, i, P^k_n) \leq k \) for each \( i = 1 \) to \( D = diam(P^k_n) \). If there are \( k \) vertices of \( P^k \) in \( T^k_n \) adjacent to \( u \), then \( p(u, 1, T^k_n) \geq p(v, 1, P^k_n) \). If not, \( u \) will be a universal vertex of \( T^k_n \) (that is, a vertex adjacent to all the other vertices of \( T^k_n \)). Thus in any case, \( p(u, 1, T^k_n) \geq p(v, 1, P^k_n) \).

If \( p(u, 2, T^k_n) < p(v, 2, P^k_n) \leq k \), then \( diam(T^k_n) \leq 2 \) (This is because if \( diam(T^k_n) \geq 3 \), then along the longest path in \( T^k_n \), there will be \( k \) vertices which would be at distance 2 from \( u \) which is a contradiction). This gives \( p(u, 1, T^k_n) + p(u, 2, T^k_n) = (n - 1) \geq p(v, 1, P^k_n) + p(u, 2, P^k_n) + p(v, 3, P^k_n) \), and as \( 7 > 4 \), \( 7p(u, 1, T^k_n) + 4p(u, 2, T^k_n) \geq 7p(v, 1, P^k_n) + 4p(v, 2, P^k_n) + p(v, 3, P^k_n) \).

Next if, \( p(u, 2, T^k_n) \geq p(v, 2, P^k_n) \) and \( p(u, 3, T^k_n) \geq p(v, 3, P^k_n) \), then clearly, \( B \geq C \). Otherwise, \( diam(T^k_n) \leq 3 \). (Same argument as above) which shows that \( p(u, 1, T^k_n) + p(u, 2, T^k_n) + p(u, 3, T^k_n) = (n - 1) \geq p(v, 1, P^k_n) + p(v, 2, P^k_n) + p(v, 3, P^k_n) \) and hence \( 7p(u, 1, T^k_n) + 4p(u, 2, T^k_n) + p(u, 3, T^k_n) \geq 7p(v, 1, P^k_n) + 4p(v, 2, P^k_n) + p(v, 3, P^k_n) \).

It can easily be seen from the proof of Theorem 2.2 that when \( k = 1 \), we have strict inequality for \( n \geq 5 \).
**Corollary 2.3.** If $G$ is a connected graph of order $n$, then $W(\mu(G^k)) \leq W(\mu(P_n^k))$.

**Proof.** Let $T$ be a spanning tree of $G$. In view of Theorem 2.2, it suffices to prove that $W(\mu(G^k)) \leq W(\mu(T^k))$. Any pair of vertices of $T^k$ at distance $i$ will be at distance at most $i$ in $G^k$. Therefore, $7p(1,G^k) + 4p(2,G^k) + p(3,G^k) \geq 7p(1,T^k) + 4p(2,T^k) + p(3,T^k)$. Thus $W(\mu(G^k)) \leq W(\mu(P_n^k))$. ■

3. **NG Type Results for the Wiener Number of Mycielski Graphs and Their Powers**

When $G$ (of order $n$ and size $m$) has no isolated vertices, $\mu(G)$ is connected while $\mu(G)$ is connected always. It is easy to see that the diameter of $\mu(G)$ is 2 and one can establish that $W(\mu(G)) = 2n^2 + 2n + 3m$.

This shows that $W(\mu(G)) + W(\mu(G)) = 8n^2 + n - 4m - 4p(2,G) - p(3,G).

As in the proof of Theorem 2.2, one can prove the following.

**Theorem 3.1.** $W(\mu(S_n^k)) + W(\mu(S_n^k)) \leq W(\mu(T_n^k)) + W(\mu(T_n^k)) \leq W(\mu(P_n^k)) + W(\mu(P_n^k))$ for any $k \geq 1$.

Now $W(\mu(G)) + W(\mu(G))$ is maximum, when $4m + 4p(2,G) + p(3,G)$ is least. As $W(\mu(P_n^k)) = \sum_{i=1}^{n-1} [\frac{k}{k}](n-i)$ (see [1]), $p(i, P_n^k) = \sum_{j=1}^{n} [n-(k(i-1)+j)]$ for $i < D$, the diameter of $P_n^k$ and thus we see that $4m + 4p(2,P_n^k) + p(3,P_n^k)$ is least when $k = 1$. From the proof of Corollary 2.3, $W(\mu(G^k)) + W(\mu(G^k)) \leq W(\mu(T^k)) + W(\mu(T^k))$ where $T$ is a spanning tree of $G$. Hence, for $n \geq 3$, we have $W(\mu(G^k)) + W(\mu(G^k)) \leq W(\mu(P_n^k)) + W(\mu(P_n^k)) \leq W(\mu(P_n^k)) + W(\mu(P_n^k)) = 8n^2 - 8n + 15$. $W(\mu(G)) + W(\mu(G))$ is minimum for graphs with diameter at most two and for these graphs $W(\mu(G)) + W(\mu(G)) = 8n^2 - 8n + 15$. Zhang and Wu [15] presented the NG type inequality for the Wiener number as given in Theorem 1.2. In our case, for Mycielski graphs $|V(\mu(G))| = 2n + 1$. Thus the corresponding inequality of Zhang and Wu [15] for graphs of order $2n + 1$ is given by $6n^2 + 3n \leq W(\mu(G)) + W(\mu(G)) \leq W(\mu(G)) + W(\mu(G))$. We can easily see that our bound for $W(\mu(G^k)) + W(\mu(G^k))$ is better than the bound of Zhang and Wu for $\mu(G^k)$ as $\frac{8n^2 + 24n^2 + 22n}{6} - (8n^2 - 8n + 15) > 0$, $n \geq 3$. 


In a similar way, we might be tempted to obtain the NG type inequalities for the following sums:

(i) \( W(\mu(G)^k) + W(\bar{\mu}(G)^k) \),
(ii) \( W(\mu(G)^k) + W(\bar{\mu}(G)^k) \),
(iii) \( W(\mu(G^k)) + W(\mu(\bar{G}^k)) \),
(iv) \( W(\mu(G^k)) + W(\mu(\bar{G}^k)) \).

Of these four, (i), (ii) and (iii) are uninteresting as \( \bar{G}^k \) is disconnected in most of the choices for \( G \) while \( \mu(G)^k \) \((k \geq 2)\) is always disconnected (as \( u \) becomes a universal vertex in \( (\mu(G))^k \)) and diameter of \( \mu(G) \) and \( \bar{\mu}(G) \) are 4 and 2 respectively. Thus NG type inequality seems interesting only for (iv). For this, we need the following lemma due to Zhang and Wu [15].

**Lemma 3.2.** Let \( G \) be a connected graph with connected complement. Then

1. if \( \text{diam}(G) > 3 \), then \( \text{diam}(\bar{G}) = 2 \),
2. if \( \text{diam}(G) = 3 \), then \( \bar{G} \) has a spanning subgraph which is a double star (see Figure 3.1).

![Figure 3.1](image)

Let \( G \) be a graph of order \( n \geq 5 \) with connected complement \( \bar{G} \). If \( \text{diam}(\bar{G}) = 2 \), we can observe the following.

(i) \( p(2, \bar{G}) = p(1, G) \).
(ii) \( W(\mu(\bar{G})) = 6n^2 - n - 7(\binom{n}{2} + p(2, \bar{G})) - 4p(2, \bar{G}) = \frac{5}{2}n^2 + \frac{3}{2}n + 3p(1, G) \).
(iii) \( W(\mu(G)) + W(\mu(\bar{G})) = \frac{15}{2}n^2 + \frac{3}{2}n - 4p(1, G) - 4p(2, G) - p(3, G) \). \hspace{1cm} (3.1)

For \( k \geq 2 \), \( G^k = P_n^k = K_n \) which implies that \( \mu(G^k) = \mu(P_n^k) \). Therefore, by virtue of Corollary 2.3, we get that \( W(\mu(G^k)) + W(\mu(\bar{G}^k)) \leq W(\mu(P_n^k)) + \)
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$W(\mu(P_n^k))$ for $k \geq 2$. The above inequality also holds for $k = 1$. This could be seen by arguments similar to those given in the proof of Theorem 2.2 and Corollary 2.3. Thus we have,

**Theorem 3.3.** Let $G$ be a connected graph of order $n \geq 5$ with connected complement $\overline{G}$. If $\text{diam}(G) = 2$, then $W(\mu(G^k)) + W(\mu(\overline{G}^k)) \leq W(\mu(P_n^k)) + W(\mu(\overline{P}_n^k))$.

**Lemma 3.4.** Let $G$ be a connected graph of order $n \geq 5$ with connected complement $\overline{G}$. Then $W(\mu(G^2)) + W(\mu(\overline{G}^2)) \leq W(\mu(P_n^2)) + W(\mu(\overline{P}_n^2))$.

**Proof.** As $\text{diam}(\overline{P}_n^2) = 2$, by using Theorem 2.1,

$$W(\mu(\overline{P}_n^2)) = 6n^2 - n - 7p(1, \overline{P}_n^2)$$

$$= 6n^2 - n - 7\left(\binom{n}{2}\right) = \frac{5}{2}n^2 + \frac{5}{2}n.$$  

For $n = 5$, $W(\mu(P_5^2)) = 6.25 - 5 - 7(4 + 3) - 4(2 + 1) = 84.$

For $n \geq 6$, $W(\mu(P_n^2)) = 6n^2 - n - 7p(1, P_n^2) - 4p(2, P_n^2) - p(3, P_n^2)$

$$= 6n^2 - n - 14n + 21 - 8n + 28 - 2n + 11$$

$$= 6n^2 - 25n + 60.$$  

Hence, $W(\mu(P_n^2)) + W(\mu(\overline{P}_n^2)) = 159$, and

$$W(\mu(P_n^2)) + W(\mu(\overline{P}_n^2)) = \frac{17}{2}n^2 - \frac{45}{2}n + 60,$$  

for $n \geq 6$.

By virtue of Theorem 3.3, it is enough to consider the case when, $\text{diam}(G) = \text{diam}(\overline{G}) = 3$. For these $G$ and $\overline{G}$, $p(1, G) = p(2, G) + p(3, G)$, $p(1, \overline{G}) = p(2, G) + p(3, G)$ and $p(1, G) + p(1, \overline{G}) = \left(\binom{n}{2}\right)$. Now by Theorem 2.1,

$$W(\mu(G^2)) = 6n^2 - n - 7p(1, G^2) - 4p(2, G^2)$$

$$= 6n^2 - n - 7(p(1, G) + p(2, G)) - 4p(3, G)$$

$$= 6n^2 - n - 7(p(1, G) - 7(p(1, \overline{G}) - p(3, G)) - 4p(3, G)$$

$$= 6n^2 - n - 7\left(\binom{n}{2}\right) + 3p(3, G).$$
Thus, \( W(\mu(G^2)) + W(\mu(G')) = 12n^2 - 2n - 7n^2 + 7n + 3(p(3,G) + p(3,G')) \),

\[
(3.3) \quad W(\mu(G^2)) + W(\mu(\overline{G}^2)) = 5n^2 + 5n + 3(p(3,G) + p(3,\overline{G})).
\]

As \( \text{diam}(G) = \text{diam}(\overline{G}) = 3 \), by Lemma 3.2 each of \( G \) and \( \overline{G} \) contains a double star, say, \( S_{a_1,b_1} \) and \( S_{a_2,b_2} \) (see Figure 3.1) as spanning subgraphs of \( G \) and \( \overline{G} \) respectively. Hence \( p(3,G) \leq (a_1 - 1)(b_1 - 1) = a_1b_1 - n + 1 \) and \( p(3,\overline{G}) \leq (a_2 - 1)(b_2 - 1) = a_2b_2 - n + 1 \). Also, \( a_i b_i \leq \lfloor \frac{n^2}{4} \rfloor \) for \( i = 1, 2 \). Thus,

\[
(3.4) \quad W(\mu(G^2)) + W(\mu(\overline{G}^2)) \leq 5n^2 - n + 6\lfloor \frac{n^2}{4} \rfloor + 6.
\]

It can be seen that \( 5n^2 - n + 6\lfloor \frac{n^2}{4} \rfloor + 6 < \frac{17}{2} n^2 - \frac{45}{2} n + 60 \), for \( n \geq 7 \). We now consider the remaining cases, namely 5 and 6 separately.

**Case (i).** \( n = 5 \).

When \( n = 5 \), by equations (3.2) and (3.3), \( W(\mu(G^2)) + W(\mu(\overline{G}^2)) = 125 + 25 + 3(p(3,G) + p(3,\overline{G})) \leq 162 \) and we have already seen that, \( W(\mu(P_5^2)) + W(\mu(\overline{P_5}^2)) = 159 \). We show that \( W(\mu(G^2)) + W(\mu(\overline{G}^2)) \leq 159 \). Suppose \( W(\mu(G^2)) + W(\mu(\overline{G}^2)) = 160 \), then \( p(3,G) + p(3,\overline{G}) = \frac{10}{3} \), which is a contradiction. Similarly, we will have a contradiction when \( W(\mu(G^2)) + W(\mu(\overline{G}^2)) = 161 \). Finally, if \( W(\mu(G^2)) + W(\mu(\overline{G}^2)) = 162 \), then \( p(3,G) + p(3,\overline{G}) = \frac{12}{3} = 4 \). Since \( n = 5 \) and \( \text{diam}(G) = \text{diam}(\overline{G}) = 3 \), \( p(3,G) \) and \( p(3,\overline{G}) \) cannot be greater than 2 and therefore \( p(3,G) = p(3,\overline{G}) = 2 \). There are only two graphs \( G \) of order 5 (see Figure 3.2) with the property that \( n = 5, p(3,G) = 2 \). But for these two graphs \( p(3,\overline{G}) = 0 \) which is a contradiction.

![Fig 3.2](image-url)
Case (ii). \( n = 6 \).
Here \( W(\mu(G^2)) + W(\mu(G^2)) = 210 + 3(p(3, G) + p(3, \overline{G})) \leq 234 \) and \( W(\mu(P_6^2)) + W(\mu(\overline{P}_6^2)) = 231 \). Proving \( W(\mu(G^2)) + W(\mu(G^2)) \leq 231 \) is similar to case (i). In this case the graphs with the required property are as shown in Figure 3.3.

\[
\begin{align*}
G: & \quad \begin{array}{c}
\begin{array}{cccc}
6 \quad 1 \quad 2 \quad 3 \quad 4
\end{array} \\
\begin{array}{cccc}
1 \quad 2 \quad 3 \quad 4
\end{array}
\end{array} \\
\overline{G}: & \quad \begin{array}{c}
\begin{array}{cccc}
6 \quad 5 \quad 4 \quad 3 \quad 2
\end{array} \\
\begin{array}{cccc}
2 \quad 3 \quad 4 \quad 5
\end{array}
\end{array}
\end{align*}
\]

\text{Fig 3.3}

We now give the result for a general \( k \).

**Theorem 3.5.** Let \( G \) be a connected graph of order \( n \geq 5 \) with connected complement \( \overline{G} \). Then for any \( k \geq 1 \), \( 5n^2 + 5n \leq W(\mu(G^k)) + W(\mu(\overline{G}^k)) \leq W(\mu(P_n^k)) + W(\mu(P_n^k)) = \frac{17}{2}n^2 - \frac{15}{2}n + 15 \).

**Proof.** \( W(\mu(G^k)) + W(\mu(\overline{G}^k)) \) is minimum when \( G^k \) and \( \overline{G}^k \) are complete. Thus \( 5n^2 + 5n \leq W(\mu(G^k)) + W(\mu(\overline{G}^k)) \). By equation 3.1 and arguments similar to that in Theorem 2.2, \( W(\mu(G)) + W(\mu(\overline{G})) \leq W(\mu(P_n)) + W(\mu(P_n)) \). By virtue of Theorem 3.3 and Lemma 3.4, the only case left out for the upper bound to be true is when \( \text{diam}(G) = \text{diam}(\overline{G}) = 3 \) and \( k \geq 3 \). In this case, \( G^k = \overline{G}^k = K_n \) and we see that \( W(\mu(G^k)) \) is minimum for \( G^k = K_n \) and therefore \( W(\mu(G^k)) + W(\mu(\overline{G}^k)) \leq W(\mu(P_n^k)) + W(\mu(P_n^k)) \leq W(\mu(P_n)) + W(\mu(P_n)) = \frac{17}{2}n^2 - \frac{15}{2}n + 15 \) (by using equation 3.1).

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