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ON THE STRUCTURE AND ZERO DIVISORS OF THE CAYLEY-DICKSON SEDENION ALGEBRA

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Abstract

The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions), and $\mathbb{O}$ (octonions) are real division algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. By doubling $\mathbb{R}$ (dim 1), we obtain $\mathbb{C}$ (dim 2), then $\mathbb{C}$ produces $\mathbb{H}$ (dim 4), and $\mathbb{H}$ yields $\mathbb{O}$ (dim 8). The next doubling process applied to $\mathbb{O}$ then yields an algebra $\mathbb{S}$ (dim 16) called the *sedenions*. This study deals with the subalgebra structure of the *sedenion algebra* $\mathbb{S}$ and its *zero divisors*. In particular, it shows that $\mathbb{S}$ has subalgebras isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and a newly identified algebra $\widetilde{\mathbb{O}}$ called the *quasi-octonions* that contains the zero-divisors of $\mathbb{S}$.

Keywords: sedenions, subalgebras, zero divisors, octonions, quasi-octonions, quaternions, Cayley-Dickson process, Fenyves identities.

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1. Introduction

In the past, non-associative algebras and related structures with *zero divisors* have not been given much attention because they did not appear to have any "useful" applications in most mathematical disciplines. Lately, however, a lot of attention has been focused by theoretical physicists on the Cayley-Dickson algebras $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) because of their increasing usefulness in formulating many of the new theories of elementary particles. In particular, the octonions $\mathbb{O}$ (which is the only non-associative normed division algebra over the reals; see [1], [8]) has been found to be involved in so many unexpected places (like *string theory, quantum theory, Clifford algebras, topology, etc.*).

These algebras are obtained by a doubling procedure called the *Cayley-Dickson Process (CDP)*. By doubling the real numbers $\mathbb{R}$ (dim $2^0 = 1$) we obtain the complex numbers $\mathbb{C}$ (dim $2^1 = 2$), then $\mathbb{C}$ produces the quaternions $\mathbb{H}$ (dim $2^2 = 4$), and $\mathbb{H}$ yields $\mathbb{O}$ (dim $2^3 = 8$), all of which are normed division algebras. The next doubling process applied to $\mathbb{O}$ then yields an algebra $\mathbb{S}$ (dim $2^4 = 16$) called the *sedenion algebra*. This doubling process can be extended beyond the sedenions to form what are known as the *2^n-ions* (see [5], [7]).

The problem with CDP is that each step of the doubling process results in a progressive loss of structure. Thus, $\mathbb{R}$ is an ordered field with all the nice properties we are so familiar with in dealing with numbers like: the *division property, associative property, commutative property, self-conjugate property*, etc. When we double $\mathbb{R}$ to obtain $\mathbb{C}$, it loses the self-conjugate property (and is no longer an ordered field), next $\mathbb{H}$ loses the commutative property, and $\mathbb{O}$ loses the associative property. Finally, when we double $\mathbb{O}$ to obtain $\mathbb{S}$, it loses the division property. This means that $\mathbb{S}$ is non-commutative, non-associative, and is not a division algebra because it has *zero divisors*. No wonder, most mathematicians shy away from the sedenions and some even consider $\mathbb{S}$ as a "pathological" case ([6]).

The captivating thing about $\mathbb{S}$ is that all of the real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ fit nicely inside it as subalgebras. Hence, any object involving these algebras can be dealt with in $\mathbb{S}$. Moreover, we see that $\mathbb{S}$ is the double of $\mathbb{O}$ which has found several applications in theoretical physics and related fields (see [1], [8], and [14]). So several theorists have found it reasonable to ask (cf. [13]):

If the octonions are so good, would not the sedenions be even better?

2. The sedenions

The Cayley-Dickson sedenion algebra $\mathbb{S}$ is often defined as a non-commutative, non-associative, non-alternative, but power-associative 16 dimensional algebra with a quadratic form and whose elements are constructed from real numbers $\mathbb{R}$ by iterations of the Cayley-Dickson Process (see [5]). Moreover, it is neither a composition algebra nor a division algebra because it has zero divisors.

Let $E_{16} = \{\mathbf{e}_i \in \mathbb{S} \mid i = 0, 1, ..., 15\}$ be the *canonical basis* of $\mathbb{S}$, where $\mathbf{e}_0$ is the *unit* (or *identity*) and $\mathbf{e}_1, ..., \mathbf{e}_{15}$ are called *imaginaries*. Then every sedenion $\mathbf{a} \in \mathbb{S}$ can be expressed as a linear combination of the *base elements* $\mathbf{e}_i \in E_{16}$, that is,

$$\mathbf{a} = \sum_{i=0}^{15} a_i \mathbf{e}_i = a_0 + \sum_{i=1}^{15} a_i \mathbf{e}_i$$

where $a_i \in \mathbb{R}$. Here a_0 is called the *real part* of $\mathbf{a}$ while $\sum_{i=1}^{15} a_i \mathbf{e}_i$ is called its *imaginary part*.

Addition of sedenions is done component-wise. On the other hand, multiplication is defined by *bilinearity* and the multiplication rule of the base elements. Thus, if $\mathbf{a}, \mathbf{b} \in \mathbb{S}$, we have:

$$\mathbf{ab} = \left(\sum_{i=0}^{15} a_i \mathbf{e}_i\right)\left(\sum_{j=0}^{15} b_j \mathbf{e}_i\right) = \sum_{i,j=0}^{15} a_i b_j (\mathbf{e}_i \mathbf{e}_j) = \sum_{i,j,k=1}^{15} f_{ij} \gamma_{ij}^k \mathbf{e}_k$$

where $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \in E_{16}$, $f_{ij} = a_i b_j \in \mathbb{R}$, and the quantities $\gamma_{ij}^k \in \mathbb{R}$ are called *structure constants*. The multiplication rule of the sedenion base elements is given by

$$\mathbf{e}_i \mathbf{e}_j = \sum_{k=0}^{15} \gamma_{ij}^k \mathbf{e}_k$$

and is summarized in Table 1.

Since $\mathbb{S}$ is the double of $\mathbb{O}$, it contains $\mathbb{O}$ as a subalgebra. Thus, the indices $i = 0, 1, ..., 7$ correspond to the octonion base elements, while those where $i = 8, ..., 15$ correspond to the pure sedenion base elements. Moreover, $\mathbb{O}$ is the double of $\mathbb{H}$, and $\mathbb{H}$ is the double of $\mathbb{C}$. Hence, $\mathbb{H}$ and $\mathbb{C}$ are also subalgebras of $\mathbb{S}$. This is nicely shown by the broken lines in Table 1.

Table 1. Multiplication table of the sedenion base elements. For simplicity, the entries in this table are the indices of the base elements, that is, we have set $i \equiv \mathbf{e}_i$, where $i = 0, 1, ..., 15$.

	*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
C	1	1	−0	3	−2	5	−4	−7	6	9	−8	−11	10	−13	12	15	−14
	2	2	−3	−0	1	6	7	−4	−5	10	11	−8	−9	−14	−15	12	13
H	3	3	2	−1	−0	7	−6	5	−4	11	−10	9	−8	−15	14	−13	12
	4	4	−5	−6	−7	−0	1	2	3	12	13	14	15	−8	−9	−10	−11
	5	5	4	−7	6	−1	−0	−3	2	13	−12	15	−14	9	−8	11	−10
	6	6	7	4	−5	−2	3	−0	−1	14	−15	−12	13	10	−11	−8	9
O	7	7	−6	5	4	−3	−2	1	−0	15	14	−13	−12	11	10	−9	−8
	8	8	−9	−10	−11	−12	−13	−14	−15	−0	1	2	3	4	5	6	7
	9	9	8	−11	10	−13	12	15	−14	−1	−0	−3	2	−5	4	7	−6
	10	10	11	8	−9	−14	−15	12	13	−2	3	−0	−1	−6	−7	4	5
	11	11	−10	9	8	−15	14	−13	12	−3	−2	1	−0	−7	6	−5	4
	12	12	13	14	15	8	−9	−10	−11	−4	5	6	7	−0	−1	−2	−3
	13	13	−12	15	−14	9	8	11	−10	−5	−4	7	−6	1	−0	3	−2
	14	14	−15	−12	13	10	−11	8	9	−6	−7	−4	5	2	−3	−0	1
	15	15	14	−13	−12	11	10	−9	8	−7	6	−5	−4	3	2	−1	−0

The above multiplication rule can also be expressed more compactly by means of 35 *associative triples* (or *cycles*). These are listed below in two sets: *octonion triplets* and *sedenion triplets*.

OCTONION TRIPLETS:

(1,2,3), (1,4,5), (1,7,6), (2,4,6), (2,5,7), (3,4,7), (3,6,5)

SEDENION TRIPLETS:

(1,8,9), (1,11,10), (1,13,12), (1,14,15)

(2,8,10), (2,9,11), (2,14,12), (2,15,13)

(3,8,11), (3,10,9), (3,15,12), (3,13,14)

(4,8,12), (4,9,13), (4,10,14), (4,11,15)

(5,8,13), (5,12,9), (5,10,15), (5,14,11)

(6,8,14), (6,15,9), (6,12,10), (6,11,13)

(7,8,15), (7,9,14), (7,13,10), (7,12,11)

If (a, b, c) is any given triplet, then $ab = c$ and $ba = -c$. This is also true of any cyclic permutation of a, b, c; e.g., $bc = a$ and $cb = -a$, etc. Moreover, given the triplet (a, b, c), then $(ab)c = a(bc)$. Similarly, this is also true of any cyclic permutation of a, b, c; e.g., $(bc)a = b(ca)$, etc. The elements in a triplet therefore associate and anti-commute.

Remark 1. There are several 16-dimensional algebras or ”semi-algebras” that are now called “*sedenions*” in the literature. One of these is the *Conway-Smith sedenions* ([4]) which is a semi-algebra with a multiplicative norm and is thus different from the *Cayley-Dickson sedenions* discussed in this paper. Another one is that defined by J.D.H. Smith in [11] that is also a semi-algebra with a multiplicative norm: it contains the octonions as a subalgebra.

3. SUBALGEBRAS OF THE SEDENIONS

The structure of the sedenion algebra is determined primarily by its subalgebra composition. As noted earlier, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ fit nicely into the sedenion algebra $\mathbb{S}$ as subalgebras as a consequence of the Cayley-Dickson Process. Thus we find from Table 1 that $\mathbb{S}$ contains $\mathbb{O}$, which contains $\mathbb{H}$, which contains $\mathbb{C}$, and which finally contains $\mathbb{R}$ as subalgebras. In addition, any other subalgebras of $\mathbb{O}$, $\mathbb{H}$, and $\mathbb{C}$ are also subalgebras of $\mathbb{S}$.

Every finite dimensional algebra (see [10]) is basically defined by the *multiplication rule* of its basis E_n. It can be shown that the set $E_{16} = \{\mathbf{e}_i \mid i = 0, 1, ..., 15\}$ of 16 sedenion base elements generates a set $\mathbf{S}_L = \{\pm\mathbf{e}_i \mid i = 0, 1, ..., 15\}$ of order 32, where $\mathbf{e}_0$ is the identity element, that forms a *non-commutative* loop under sedenion multiplication. This loop $\mathbf{S}_L$, which we shall call the *Cayley-Dickson sedenion loop*, is embedded (see [8]) in the sedenion space and its subloops determine the *basic subalgebras* of $\mathbb{S}$ (the subalgebras generated by the base elements of $\mathbb{S}$).

To determine the properties and basic subalgebras of $\mathbb{S}$, we must therefore analyze this embedded loop $\mathbf{S}_L$ by decomposing it into its subloops and identifying each of them. We do this by means of the software *FINITAS* ([9]) – a computer program for the analysis and construction of finite algebraic structures. The results of this analysis are as follows:

I. The non-commutative loop $\mathbf{S}_L$ belongs to the class of *non-associative finite invertible loops* (NAFIL). Analysis shows that it satisfies the following properties:

- PAP (*Power Associative Property*), IP (*L/R Inverse Property*), WIP (*Weak Inverse Property*), AAIP (*Antiautomorphic Inverse Property*), SAIP (*L/R Semiautomorphic Inverse Property*), AP (*L/R Alternative Property*), FL (*Flexible Law*), RIF *Loop property*, CL (*C-Loop Property*), and NSLP (*Nuclear Square Loop Property*: LN, MN, RN). Moreover, it follows from PAP, IP, AP, and FL that it is also diassociative. (See Table 2.)

- All elements of $\mathbf{S}_L$, except e_0 and $-e_0$, are of order 4; its center is $\{e_0, -e_0\}$; and all squares are in this center.

II. The loop $\mathbf{S}_L$ has exactly 67 subloops (numbered 1 to 67 by *FINITAS*), 66 of which are non-trivial and normal. The block diagram of the lattice of these subloops is shown in Figure 1.

- There are 15 subloops of order 16. All of these are non-abelian NAFILs of two types: (a) eight NAFILs isomorphic to the *octonion loop* $\mathbf{O}_L$ (the Moufang loop generated by the basis of $\mathbb{O}$), and (b) seven NAFILs that are isomorphic to a loop which we shall call the *quasi-octonion loop* $\widetilde{\mathbf{O}}_L$. This loop is not isomorphic to the octonion loop $\mathbf{O}_L$ because it does not satisfy the Moufang identity. These are the maximal subloops of $\mathbf{S}_L$.

- There are 35 subloops of order 8. All of these are non-abelian groups isomorphic to the quaternion group $\mathbf{Q}$ of order 8. These are subloops of the copies of $\mathbf{O}_L$ and $\widetilde{\mathbf{O}}_L$.

- There are 15 subloops of order 4. All of these are abelian groups isomorphic to the cyclic group $\mathbf{C}_4$ of order 4. These are subloops of the copies of $\mathbf{Q}$.

- There is only one (1) subloop of order 2. This is a group isomorphic to the cyclic group $\mathbf{C}_2$ of order 2 and is the center of $\mathbf{S}_L$.

- There is only one (1) subloop of order 1. This is the trivial group.

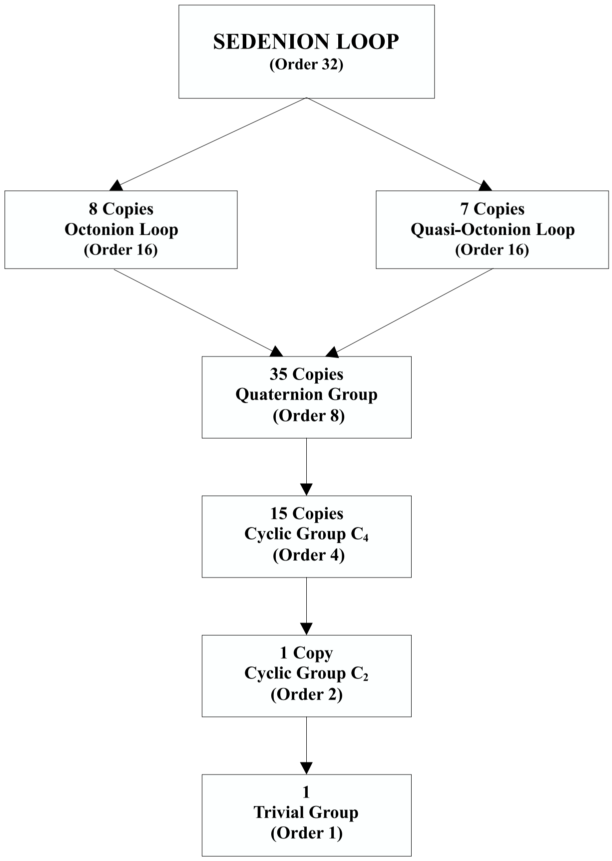

Figure 1. The lattice diagram (in block form) of the subloop structure of the sedenion loop $\mathbf{S}_L$ of order 32.

It follows from the above subloop analysis that:

Proposition 1. *Every non-trivial subloop of* $\mathbf{S}_L$ *is isomorphic to one of the following loops:* $\mathbf{O}_L$, $\widetilde{\mathbf{O}}_L$, $\mathbf{Q}$, $\mathbf{C}_4$, *and* $\mathbf{C}_2$. ■

The subloops of $\mathbf{S}_L$ that are isomorphic to $\mathbf{O}_L$, $\widetilde{\mathbf{O}}_L$, $\mathbf{Q}$, $\mathbf{C}_4$, and $\mathbf{C}_2$ shall be called *copies* of these loops. The loop $\widetilde{\mathbf{O}}_L$ represents the class of 7 isomorphic subloops of $\mathbf{S}_L$ (numbers 4, 7, 10, 18, 21, 29, 32) listed in Table 3. Note that in each of the octonion and quasi-octonion copies, the first three of the 7 imaginaries are elements of an octonion triplet while the remaining four are pure sedenion base elements. These imaginaries have very special properties.

The lattice of the subloops of S_L shown in Figure 1 also shows that the copies of $\mathbf{O}_L$ and $\widetilde{\mathbf{O}}_L$ contain only copies of $\mathbf{Q}$, $\mathbf{C}_4$, and $\mathbf{C}_2$ as subloops. This means that both $\mathbf{O}_L$ and $\widetilde{\mathbf{O}}_L$ contain only subloops isomorphic to $\mathbf{Q}$, $\mathbf{C}_4$, and $\mathbf{C}_2$. Thus, they have the same subloop composition.

To determine the important properties of the loops S_L, $\mathbf{O}_L$ and $\widetilde{\mathbf{O}}_L$, we analyzed their Cayley tables by means of the software FINITAS. The identified properties are listed in Table 2.

Table 2. This table shows some of the known loop identities satisfied by the sedenion loop $\mathbf{S}_L$, octonion loop $\mathbf{O}_L$, and the quasi-octonion loop $\widetilde{\mathbf{O}}_L$.

PROPERTY	DEFINING IDENTITY	$\mathbf{S}_L$	$\mathbf{O}_L$	$\widetilde{\mathbf{O}}_L$
IP	LIP: $x^{-1}(xy) = y$ and RIP: $(yx)x^{-1} = y$	YES	YES	YES
FL	$x(yx) = (xy)x$	YES	YES	YES
AP	LAP: $x(xy) = (xx)y$ and RAP: $x(yy) = (xy)y$	YES	YES	YES
CL	$x(y(yz)) = ((xy)y)z \rightarrow$ LC and RC	YES	YES	YES
LC	$(xx)(yz) = (x(xy))z$	YES	YES	YES
RC	$x((yz)z) = (xy)(zz)$	YES	YES	YES
MP	$(xy)(zx) = (x(yx))z$	$\times$	YES	$\times$
PAP	$x^a x^b = x^{a+b}$	YES	YES	YES
WIP	$x(yx)^{-1} = y^{-1}$	YES	YES	YES
AAIP	$(xy)^{-1} = y^{-1}x^{-1}$	YES	YES	YES
RIF	$(xy)(z(xy)) = ((x(yz))x)y$	YES	YES	YES
NSLP	LN, MN, RN	YES	YES	YES
LN	$(xx)(yz) = ((xx)y)z$	YES	YES	YES
MN	$x((yy)z) = (x(yy))z$	YES	YES	YES
RN	$x(y(zz)) = (xy)(zz)$	YES	YES	YES

We therefore see that the sedenion, octonion, and quasi-octonion loops share the same properties. This follows from the fact that a subloop $\overline{L}$ of any loop L satisfies all properties of L. Although the octonion loop $\mathbf{O}_L$ also shares all of the properties of $\widetilde{\mathbf{O}}_L$ and $\mathbf{S}_L$, it satisfies in addition the Moufang Property (MP). Hence, $\mathbf{O}_L$ and $\widetilde{\mathbf{O}}_L$ are not isomorphic.

Table 3. Subloops of the sedenion loop $\mathbf{S}_L$that are copies of $\mathbf{O}_L$ and $\widetilde{\mathbf{O}}_L$. Here, the subloop $2.\{0, 1, 2, 3, 4, 5, 6, 7, -0, -1, -2, -3, -4, -5, -6, -7\}$ corresponds to $\mathbf{O}_L$ as a consequence of the Cayley-Dickson Process.

Copies of Octonion Loop $\mathbf{O}_L$	Copies of Quasi-Octonion Loop $\widetilde{\mathbf{O}}_L$
2.{**0,1,2,3,4,5,6,7**,-0,-1,-2,-3,-4,-5,-6,-7}$\longrightarrow \mathbf{O}_L$	
3.{**0,1,2,3,8,9,10,11**,-0,-1,-2,-3,-8,-9,-10,-11}	4.{**0,1,2,3,12,13,14,15**,-0,-1,-2,-3,-12,-13,-14,-15}
6.{**0,1,4,5,8,9,12,13**,-0,-1,-4,-5,-8,-9,-12,-13}	7.{**0,1,4,5,10,11,14,15**,-0,-1,-4,-5,-10,-11,-14,-15}
9.{**0,1,6,7,8,9,14,15**,-0,-1,-6,-7,-8,-9,-14,-15}	10.{**0,1,6,7,10,11,12,13**,-0,-1,-6,-7,-10,-11,-12,-13}
17.{**0,2,4,6,8,10,12,14**,-0,-2,-4,-6,-8,-10,-12,-14}	18.{**0,2,4,6,9,11,13,15**,-0,-2,-4,-6,-9,-11,-13,-15}
20.{**0,2,5,7,8,10,13,15**,-0,-2,-5,-7,-8,-10,-13,-15}	21.{**0,2,5,7,9,11,12,14**,-0,-2,-5,-7,-9,-11,-12,-14}
28.{**0,3,4,7,8,11,12,15**,-0,-3,-4,-7,-8,-11,-12,-15}	29.{**0,3,4,7,9,10,13,14**,-0,-3,-4,-7,-9,-10,-13,-14}
31.{**0,3,5,6,8,11,13,14**,-0,-3,-5,-6,-8,-11,-13,-14}	32.{**0,3,5,6,9,10,12,15**,-0,-3,-5,-6,-9,-10,-12,-15}

The Cayley tables of the octonion loop $\mathbf{O}_L = \{\pm \mathbf{e}_i \mid i = 0, 1, ..., 7\}$ and quasi-octonion loop $\widetilde{\mathbf{O}}_L = \{\pm \mathbf{u}_i \mid i = 0, 1, ..., 7\}$ are shown in Tables 4(A) and 4(B).

Table 4(A). Cayley table of the octonion loop $\mathbf{O}_L$ of order 16.

*	e0	e1	e2	e3	e4	e5	e6	e7	-e0	-e1	-e2	-e3	-e4	-e5	-e6	-e7
e0	e0	e1	e2	e3	e4	e5	e6	e7	-e0	-e1	-e2	-e3	-e4	-e5	-e6	-e7
e1	e1	-e0	e3	-e2	e5	-e4	-e7	e6	-e1	e0	-e3	e2	-e5	e4	e7	-e6
e2	e2	-e3	-e0	e1	e6	e7	-e4	-e5	-e2	e3	e0	-e1	-e6	-e7	e4	e5
e3	e3	e2	-e1	-e0	e7	-e6	e5	-e4	-e3	-e2	e1	e0	-e7	e6	-e5	e4
e4	e4	-e5	-e6	-e7	-e0	e1	e2	e3	-e4	e5	e6	e7	e0	-e1	-e2	-e3
e5	e5	e4	-e7	e6	-e1	-e0	-e3	e2	-e5	-e4	e7	-e6	e1	e0	e3	-e2
e6	e6	e7	e4	-e5	-e2	e3	-e0	-e1	-e6	-e7	-e4	e5	e2	-e3	e0	e1
e7	e7	-e6	e5	e4	-e3	-e2	e1	-e0	-e7	e6	-e5	-e4	e3	e2	-e1	e0
-e0	-e0	-e1	-e2	-e3	-e4	-e5	-e6	-e7	e0	e1	e2	e3	e4	e5	e6	e7
-e1	-e1	e0	-e3	e2	-e5	e4	e7	-e6	e1	-e0	e3	-e2	e5	-e4	-e7	e6
-e2	-e2	e3	e0	-e1	-e6	-e7	e4	e5	e2	-e3	-e0	e1	e6	e7	-e4	-e5
-e3	-e3	-e2	e1	e0	-e7	e6	-e5	e4	e3	e2	-e1	-e0	e7	-e6	e5	-e4
-e4	-e4	e5	e6	e7	e0	-e1	-e2	-e3	e4	-e5	-e6	-e7	-e0	e1	e2	e3
-e5	-e5	-e4	e7	-e6	e1	e0	e3	-e2	e5	e4	-e7	e6	-e1	-e0	-e3	e2
-e6	-e6	-e7	-e4	e5	e2	-e3	e0	e1	e6	e7	e4	-e5	-e2	e3	-e0	-e1
-e7	-e7	e6	-e5	-e4	e3	e2	-e1	e0	e7	-e6	e5	e4	-e3	-e2	e1	-e0

Table 4(B). Cayley table of the quasi-octonion loop $\widetilde{\mathbf{O}}_L$ of order 16.

*	u0	u1	u2	u3	u4	u5	u6	u7	-u0	-u1	-u2	-u3	-u4	-u5	-u6	-u7
u0	u0	u1	u2	u3	u4	u5	u6	u7	-u0	-u1	-u2	-u3	-u4	-u5	-u6	-u7
u1	u1	-u0	u3	-u2	-u5	u4	u7	-u6	-u1	u0	-u3	u2	u5	-u4	-u7	u6
u2	u2	-u3	-u0	u1	-u6	-u7	u4	u5	-u2	u3	u0	-u1	u6	u7	-u4	-u5
u3	u3	u2	-u1	-u0	-u7	u6	-u5	u4	-u3	-u2	u1	u0	u7	-u6	u5	-u4
u4	u4	u5	u6	u7	-u0	-u1	-u2	-u3	-u4	-u5	-u6	-u7	u0	u1	u2	u3
u5	u5	-u4	u7	-u6	u1	-u0	u3	-u2	-u5	u4	-u7	u6	-u1	u0	-u3	u2
u6	u6	-u7	-u4	u5	u2	-u3	-u0	u1	-u6	u7	u4	-u5	-u2	u3	u0	-u1
u7	u7	u6	-u5	-u4	u3	u2	-u1	-u0	-u7	-u6	u5	u4	-u3	-u2	u1	u0
-u0	-u0	-u1	-u2	-u3	-u4	-u5	-u6	-u7	u0	u1	u2	u3	u4	u5	u6	u7
-u1	-u1	u0	-u3	u2	u5	-u4	-u7	u6	u1	-u0	u3	-u2	-u5	u4	u7	-u6
-u2	-u2	u3	u0	-u1	u6	u7	-u4	-u5	u2	-u3	-u0	u1	-u6	-u7	u4	u5
-u3	-u3	-u2	u1	u0	u7	-u6	u5	-u4	u3	u2	-u1	-u0	-u7	u6	-u5	u4
-u4	-u4	-u5	-u6	-u7	u0	u1	u2	u3	u4	u5	u6	u7	-u0	-u1	-u2	-u3
-u5	-u5	u4	-u7	u6	-u1	u0	-u3	u2	u5	-u4	u7	-u6	u1	-u0	u3	-u2
-u6	-u6	u7	u4	-u5	-u2	u3	u0	-u1	u6	-u7	-u4	u5	u2	-u3	-u0	u1
-u7	-u7	-u6	u5	u4	-u3	-u2	u1	u0	u7	u6	-u5	-u4	u3	u2	-u1	-u0

Remark 2. A search of the current literature on the sedenions has shown that the quasi-octonion loop $\widetilde{\mathbf{O}}_L$ has not been previously identified [10]. This loop and its 7 copies in $\mathbb{S}$ implement some of the identities of the *Bol-Moufang type* (see [9]; also known as the *Fenyves identities*) like the CL, RC, LC, and the LN, MN, RN identities (Table 2); they are the first known non-trivial natural models of these identities.

In the above tables, the 8 positive elements, $\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_7$, of the loop $\mathbf{O}_L$ are the base elements of the 8-dimensional *octonion algebra* $\mathbb{O}$ (or *Cayley numbers*). On the other hand, the elements $\mathbf{u}_0, \mathbf{u}_1, ..., \mathbf{u}_7$, of the loop $\widetilde{\mathbf{O}}_L$ are the base elements of a newly identified 8-dimensional algebra $\widetilde{\mathbb{O}}$ which we shall call the *quasi-octonion algebra*. This means that the positive elements of the copies of $\mathbf{O}_L$ and $\widetilde{\mathbf{O}}_L$ form subalgebras isomorphic to the algebras $\mathbb{O}$ and $\widetilde{\mathbb{O}}$, respectively. Note, however, that not all of the properties of the loops $\mathbf{S}_L$, $\mathbf{O}_L$, and $\widetilde{\mathbf{O}}_L$ are inherited by the corresponding algebras $\mathbb{S}$, $\mathbb{O}$, and $\widetilde{\mathbb{O}}$ that they generate.

Similarly, the positive elements of the loops that are copies of $\mathbf{Q}$, $\mathbf{C}_4$, and $\mathbf{C}_2$ in $\mathbb{S}$ define subalgebras of the copies of $\mathbb{O}$ and $\widetilde{\mathbb{O}}$ in $\mathbb{S}$. Thus, the sedenion algebra $\mathbb{S}$ contains subalgebras isomorphic to $\mathbb{O}$, $\widetilde{\mathbb{O}}$, $\mathbb{H}$, and $\mathbb{C}$. The lattice of these basic subalgebras of $\mathbb{S}$, therefore, has the same structure as that of the subloops of $\mathbf{S}_L$ shown in Figure 1. Whether or not these are the only subalgebras of $\mathbb{S}$ (to within isomorphism) is an interesting open problem.

The quasi-octonion algebra $\widetilde{\mathbb{O}}$ has all of the known properties of the sedenion algebra $\mathbb{S}$: it is non-commutative, non-associative, non-alternative, power-associative, and has a quadratic form. And, like $\mathbb{S}$, it is neither a composition nor a division algebra because it has zero divisors. Moreover, $\widetilde{\mathbb{O}}$ has all of the known properties of the octonions $\mathbb{O}$, except the Moufang identity, and determining the details of its structure is interesting open problem.

In the next section, we will show that the existence of the seven copies of $\widetilde{\mathbb{O}}$ as subalgebras of $\mathbb{S}$ is responsible for the known zero divisors of the sedenions.

4. THE ZERO DIVISORS OF THE SEDENIONS

Algebras with zero divisors are not very popular among mathematicians because very few know what to do with these unusual objects. This is also due to the fact that most of the useful algebras we are familiar with are division algebras (like $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$) where the equation $\mathbf{ab} = \mathbf{0}$ is true iff $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

As indicated in the previous sections, the sedenion algebra $\mathbb{S}$ is not a division algebra because it has zero divisors. This means that there exist sedenions $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$ such that $\mathbf{ab} = \mathbf{0}$. *But where in the sedenion space do we find these zero divisors?* This is an important question that we will now try to settle.

Several studies have been made on the zero divisors of $\mathbb{S}$ motivated by their potential applications in theoretical physics. Often cited in the literature are the following papers [5]–[7] and [13]. Of these, only R.P.C. de Marrais has determined (by what he calls a "bottom-up" approach) the actual zero divisors of $\mathbb{S}$. The others have simply dealt with zero divisors from a theoretical standpoint (called the "top-down" approach).

The actual determination of the zero divisors of the sedenions is quite tedious and time consuming. G. Moreno has exhibited only one instance of a pair of sedenion zero divisors in his paper*. K. and M. Imaeda claimed that the zero divisors of the sedenions are confined to some *hypersurfaces*

*Moreno gave in [7] the following example of a zero divisor pair: $x = e_1 + e_{10}$ and $y = e_{15} - e_4$. Thus we find that $xy = (e_1 + e_{10})(e_{15} - e_4) = 0$.

but did not explain what these are. On the other hand, de Marrais[†] has determined exactly 84 pairs of zero divisors shown in Table 5 by "isolating underlying structures from which all complicated ZD expressions and spaces in the Sedenions must be composed..."

After studying the 84 known *zero divisor pairs* in Table 5 determined by de Marrais and the subloops of the sedenion loop $\mathbf{S}_L$ in Table 3, we now have.

Proposition 2. *The known zero divisors of the sedenion algebra* $\mathbb{S}$ *are all confined to copies in* $\mathbb{S}$ *of the quasi-octonion algebra* $\widetilde{\mathbb{O}}$.

By Proposition 1, every subloop of $\mathbf{S}_L$ is isomorphic to one of the following loops: $\mathbf{O}_L$,$\widetilde{\mathbf{O}}_L, \mathbf{Q}, \mathbf{C}_4$, and $\mathbf{C}_2$. Since $\mathbf{O}_L$ and $\mathbf{Q}, \mathbf{C}_4$, $\mathbf{C}_2$ (which are groups) define subalgebras that are division algebras, then they do not have any zero divisors. It is easy to show that the quasi-octonion algebra $\widetilde{\mathbb{O}}$ has zero divisors and, therefore, all of its copies in $\mathbb{S}$ have zero divisors. This indicates that any of the known zero divisors of $\mathbb{S}$ must belong to one of the copies of $\widetilde{\mathbb{O}}$.

To verify this, consider Table 5 which lists the 84 zero divisor pairs determined by de Marrais [6]. Each zero divisor in the pair consists of two base elements of the form $(\mathbf{o} \pm \mathbf{s})$, where $\mathbf{o}$ is an octonion base element (belonging to an octonion triplet), while $\mathbf{s}$ is a pure sedenion base element. These are presented as seven sets called "**GoTo**" lists, each based on one of the 7 octonion triplets (or **O-trip**) and a set of 7 imaginary base elements called an *automorpheme*. As can be seen in Table 5, each automorpheme consists of the positive imaginary elements of a subloop of $\mathbf{S}_L$ that is a copy of $\widetilde{\mathbf{O}}_L$. Hence, they correspond to subalgebras of $\mathbb{S}$ that are copies of $\widetilde{\mathbf{O}}$. ■

Note that in any given GoTo list, the two base elements $\mathbf{o}$ and $\mathbf{s}$ in any zero divisor $(\mathbf{o} \pm \mathbf{s})$ belong to the automorpheme of that list. Moreover, the set of four base elements in each *zero divisor pair* can be found only in the automorpheme of that GoTo list.

[†]R. de Maraais has developed a set of "Production Rules" ([6]) for determining the 84 sedenion zero divisor pairs in Table 3 that is quite simple to carry out compared to the usual methods.

Table 5. List of sedenion *zero divisor pairs.* Source: Robert de Marrais, http://arXiv.org/abs/math.GM/0011260. As in Table 1, the numerals are the indices of the base elements, that is, $i \equiv e_i$.

GoTo#1	Based on Octonion Triplet**(1,2,3)**–Automorpheme:(**1,2,3,12,13,14,15**)			
	(1+13)(2–14)	(1+14)(2+13)	(1–12)(2–15)	(1–15)(2+12)
	(2–14)(3+15)	(2+13)(3–12)	(2–15)(3–14)	(2+12)(3+13)
	(3+15)(1–13)	(3–12)(1–14)	(3–14)(1+12)	(3+13)(1+15)
GoTo#2	Based on Octonion Triplet**(1,4,5)**–Automorpheme:(**1,4,5,10,11,14,15**)			
	(1+14)(4–11)	(1+11)(4+14)	(1–15)(4–10)	(1–10)(4+15)
	(4–11)(5+10)	(4+14)(5–15)	(4–10)(5–11)	(4+15)(5+14)
	(5+10)(1–14)	(5–15)(1–11)	(5–11)(1+15)	(5+14)(1+10)
GoTo#3	Based on Octonion Triplet**(1,7,6)**–Automorpheme:(**1,7,6,10,11,12,13**)			
	(1+11)(7–13)	(1+13)(7+11)	(1–10)(7–12)	(1–12)(7+10)
	(7–13)(6+12)	(7+11)(6–10)	(7–12)(6–13)	(7+10)(6+11)
	(6+12)(1–11)	(6–10)(1–13)	(6–13)(1+10)	(6+11)(1+12)
GoTo#4	Based on Octonion Triplet**(2,4,6)**–Automorpheme:(**2,4,6,9,11,13,15**)			
	(2+15)(4–9)	(2+9)(4+15)	(2–13)(4–11)	(2–11)(4+13)
	(4–9)(6+11)	(4+15)(6–13)	(4–11)(6– 9)	(4+13)(6+15)
	(6+11)(2–15)	(6–13)(2–9)	(6–9)(2+13)	(6+15)(2+11)
GoTo#5	Based on Octonion Triplet**(2,5,7)**–Automorpheme:(**2,5,7,9,11,12,14**)			
	(2+9)(5–14)	(2+14)(5+9)	(2–11)(5–12)	(2–12)(5+11)
	(5–14)(7+12)	(5+9)(7–11)	(5–12)(7–14)	(5+11)(7+ 9)
	(7+12)(2–9)	(7–11)(2–14)	(7–14)(2+11)	(7+9)(2+12)
GoTo#6	Based on Octonion Triplet**(3,4,7)**–Automorpheme:(**3,4,7,9,10,13,14**)			
	(3+13)(4-10)	(3+10)(4+13)	(3–14)(4– 9)	(3–9)(4+14)
	(4–10)(7+9)	(4+13)(7–14)	(4–9)(7–10)	(4+14)(7+13)
	(7+ 9)(3–13)	(7–14)(3–10)	(7–10)(3+14)	(7+13)(3+ 9)
GoTo#7	Based on Octonion Triplet**(3,6,5)**–Automorpheme:(**3,6,5,9,10,12,15**)			
	(3+10)(6–15)	(3+15)(6+10)	(3–9)(6–12)	(3–12)(6+ 9)
	(6–15)(5+12)	(6+10)(5–9)	(6–12)(5–15)	(6+9)(5+10)
	(5+12)(3–10)	(5–9)(3–15)	(5–15)(3+9)	(5+10)(3+12)

Consider the zero divisor pair (2–14)(3+15) found in the GoTo#1 list of Table 5. Let us evaluate this expression as the bilinear product of two sedenions $(\mathbf{e}_2 - \mathbf{e}_{14})$ and $(\mathbf{e}_3 + \mathbf{e}_{15})$. Using the multiplication rule shown in Table 1, we have:

$$
\begin{aligned}
(\mathbf{e}_2 - \mathbf{e}_{14})(\mathbf{e}_3 + \mathbf{e}_{15}) &= \mathbf{e}_2 \cdot \mathbf{e}_3 + \mathbf{e}_2 \cdot \mathbf{e}_{15} - (\mathbf{e}_{14} \cdot \mathbf{e}_3) - (\mathbf{e}_{14} \cdot \mathbf{e}_{15}) \\
&= \mathbf{e}_1 + \mathbf{e}_{13} - \mathbf{e}_{13} - \mathbf{e}_1 = 0
\end{aligned}
$$

Thus, we find that $(\mathbf{e}_2 - \mathbf{e}_{14})(\mathbf{e}_3 + \mathbf{e}_{15}) = 0$ although neither $(\mathbf{e}_2 - \mathbf{e}_{14})$ nor $(\mathbf{e}_3 + \mathbf{e}_{15})$ is equal to zero. Therefore, the sedenions $(\mathbf{e}_2 - \mathbf{e}_{14})$ and $(\mathbf{e}_3 + \mathbf{e}_{15})$ are zero divisors. All of the zero divisor pairs listed in Table 5 can be evaluated in the same way giving the same results.

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