THE PROJECTIVE PLANE CROSSING NUMBER OF THE CIRCULANT GRAPH $C(3k; \{1, k\})$

Pak Tung Ho

Department of Mathematics, Sogang University, Seoul 121-742, Korea
e-mail: ptho@sogang.ac.kr

Abstract

In this paper we prove that the projective plane crossing number of the circulant graph $C(3k; \{1, k\})$ is $k - 1$ for $k \geq 4$, and is 1 for $k = 3$.

Keywords: crossing number, circulant graph, projective plane.

2010 Mathematics Subject Classification: 05C10.

1. Introduction

The crossing number is an important measure of the non-planarity of a graph. Bhatt and Leighton [1] showed that the crossing number of a network (graph) is closely related to the minimum layout area required for the implementation of a VLSI circuit for that network. In general, determining the crossing number of a graph is hard. Garey and Johnson [3] showed that it is NP-complete. In fact, Hliněný [6] has proved that the problem remains NP-complete even when restricted to cubic graphs. Moreover, the exact crossing number is not known even for specific graph families, such as complete graphs [16], complete bipartite graphs [11, 22], and circulant graph [8, 12, 13, 14, 20, 23]. For more about crossing number, see [2, 21] and references therein.

Attention has been paid to the crossing number of graphs on surfaces [4, 5, 7, 9, 10, 17, 18, 19]. However, exact values are known only for very restricted classes of graphs. In this paper, we compute the projective plane crossing number of the circulant graph $C(3k; \{1, k\})$.

Theorem 1. The projective plane crossing number of the circulant graph $C(3k; \{1, k\})$ is given by

$$cr_1(C(3k; \{1, k\})) = \begin{cases} k - 1 & \text{for } k \geq 4, \\ 1 & \text{for } k = 3. \end{cases}$$
Note that there are only a few infinite classes of graphs whose projective plane crossing number are known exactly. See [9, 19].

Here are some definitions. Let \( G \) be a simple graph with the vertex set \( V = V(G) \) and the edge set \( E = E(G) \). The circulant graph \( C(n; S) \) is the graph with the vertex set \( V(C(n; S)) = \{ v_i \mid 1 \leq i \leq n \} \) and the edge set \( E(C(n; S)) = \{ v_i v_j \mid 1 \leq i, j \leq n, (i-j) \mod n \in S \} \) where \( S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor \} \).

The projective plane crossing number \( cr_1(G) \) of \( G \) is the minimum number of crossings of all the drawings of \( G \) in the projective plane having the following properties: (i) no edge has a self-intersection; (ii) no two adjacent edges intersect; (iii) no two edges intersect each other more than once; (iv) each intersection of edges is a crossing rather than tangential; and (v) no three edges intersect in a common point. Similarly one can define the plane crossing number \( cr(G) \) of the graph \( G \). In a drawing \( D \), if an edge (or a set of edges) does not cross other edges, we call it clean; otherwise, we call it cross. For a drawing \( D \), the total number of crossings is denoted by \( v(D) \).

Let \( A \) and \( B \) be two (not necessary disjoint) subsets of the edge set \( E \). In a drawing \( D \), the number of crossings crossed by an edge in \( A \) and another edge in \( B \) is denoted by \( v_D(A, B) \). In particular, \( v_D(A, A) \) is denoted by \( v_D(A) \), and hence \( v(D) = v_D(E) \). By counting the number of crossings in \( D \), we have the following:

**Lemma 2.** Let \( A, B, C \) be mutually disjoint subsets of \( E \). Then,

\[
\begin{align*}
v_D(A, B \cup C) &= v_D(A, B) + v_D(A, C), \\
v_D(A \cup B) &= v_D(A) + v_D(B) + v_D(A, B).
\end{align*}
\]

The plan of this paper is as follows. In Section 2 we prove the upper bound of the projective crossing number of \( C(3k; \{1, k\}) \). In Section 3, we prove the lower bound of the projective crossing number of \( C(3k; \{1, k\}) \) by assuming Lemma 7. In Section 4, we prove Lemma 7, which says that for any drawing of \( C(3k; \{1, k\}) \) with all of its cycles being clean, its number of crossing is at least \( k - 1 \).

## 2. Upper Bounds

From now on, we will denote the circulant graph \( C(3k; \{1, k\}) \) by \( C(k) \) for simplicity. First we have the following:

**Lemma 3.** \( cr_1(C(3)) \leq 1 \).

**Proof.** One can refer to the drawing of \( C(3) \) in the projective plane in Figure 1.

\[ \blacksquare \]

**Lemma 4.** \( cr_1(C(k)) \leq k - 1 \) for \( k \geq 4 \).
**Proof.** For a non-planar graph $G$, the plane crossing number is strictly greater than the projective plane crossing number, i.e., $cr_1(G) \leq cr(G) - 1$. Lemma 4 follows from $cr(C(k)) = k$ for $k \geq 4$, which is proved in [12].

3. Lower Bounds

Next, we have the following:

**Lemma 5.** $cr_1(C(3)) \geq 1$.

**Proof.** It suffices to show that $C(3)$ cannot be embedded in the projective plane. Note that $C(3) - \{v_1v_7, v_2v_8, v_3v_6\}$ is isomorphic to $F_1(9,15)$ (see Figure 2) in the list of the minimal forbidden subgraphs for the projective plane (see Appendix A in [15]). This shows that $C(3)$ cannot be embedded in the projective plane. 

In fact, we have shown the following:

**Corollary 6.** If $e$ is an edge in the cycle $C_i$ (see the definition below) in $C(3)$, then $cr_1(C(3) - e) \geq 1$.

In $C(k)$, we define

$$C_i = \{v_{i}v_{k+i}, v_{i}v_{2k+i}, v_{k+i}v_{2k+i}\},$$

where $1 \leq i \leq k$. We have the following:

**Lemma 7.** For $k \geq 4$, let $D$ be a drawing of $C(k)$ such that $C_i$ is clean for all $1 \leq i \leq k$. Then $v(D) \geq k - 1$. 
We postpone its proof to Section 4. By assuming Lemma 7, we are in a position to prove the lower bound of $cr_1(C(k))$.

**Lemma 8.**

\[(2) \quad cr_1(C(k)) \geq k - 1 \text{ for } k \geq 4.\]

**Proof.** We will prove (2) by induction on $k$. First consider $k = 4$. Suppose $D$ is a drawing of $C(4)$. We will prove $v(D) \geq 3$ by contradiction. Suppose that $v(D) \leq 2$. Then there exists $C_i$ which crosses; otherwise, if all $C_i$ are clean, $v(D) \geq 3$ by Lemma 7.

Without loss of generality, we may assume that the edge $v_1v_5$ in $C_1$ crosses. Then there exists an edge $e$ in $D - v_1v_5$ such that $D - v_1v_5 - e$ is an embedding in the projective plane. Note that $e$ cannot be the edge in any cycle $C_1$: If $e$ is an edge in $C_1$ other than $v_1v_5$, then $D - C_1$, which is a subdivision of $C(3)$, is an embedding in the projective plane, which is impossible by Lemma 5. If $e$ is an edge in $C_i$ with $i \neq 1$, then $D - C_1 - e$, which is a subdivision of $C(3)$ minus an edge in the cycle $C_i$ is an embedding in the projective plane, which contradicts Corollary 6.

Therefore, by symmetry, we have the following possibilities: $e = v_2v_3$, $e = v_4v_5$, $e = v_5v_6$, $e = v_6v_7$, $e = v_7v_8$, $e = v_8v_9$. We will show that it is impossible for $C(4) - v_1v_5 - e$ to be embedded in the projective plane for each of these cases, which will give the required contradiction.

First, by contracting the edges $v_5v_6$ and $v_7v_8$ in $C(4) - \{v_1v_5, v_4v_5, v_8v_9\}$, we get a graph which contains a subgraph isomorphic to $F_4(10, 16)$ (see Figure 3(a)) in the list of the minimal forbidden subgraphs for the projective plane (see Appendix A in [15]). Moreover, by contracting the edges $v_3v_4$ and $v_5v_6$ in $C(4) - \{v_1v_5, v_2v_3, v_6v_7\}$, we get a graph which contains a subgraph isomorphic to $F_4(10, 16)$ (see Figure 3(b)).
Next we are going to show that $C(4) - \{v_1v_5, v_5v_6\}$ cannot be embedded in the projective plane. Suppose it is not true and let $D$ be an embedding of $C(4) - \{v_1v_5, v_5v_6\}$ in the projective plane. Delete the edge $v_2v_6$ in the drawing. Since $v_1v_5$ and $v_5v_6$ are absent, we can always draw an edge connecting $v_4$ and $v_9$ which is close to the edges $v_4v_5$ and $v_5v_9$ without producing new crossings (see Figure 4(a)). Similarly, since $v_2v_6$ and $v_5v_6$ are absent, we can draw an edge connecting $v_7$ and $v_{10}$ which is close to the edges $v_6v_7$ and $v_6v_{10}$ without producing new crossings (see Figure 4(b)). Therefore, we obtain an embedding of $C(12, \{1, 4\}) - \{v_1v_5, v_5v_6, v_2v_6\} + \{v_4v_9, v_7v_{10}\}$ in the projective plane, which is impossible since it contains a minor isomorphic to $F_5(10, 16)$ (see Figure 3(c)).

Finally, one can see that $C(12, \{1, 4\}) - \{v_1v_5, v_7v_8\}$ contains a minor isomorphic to $F_5(10, 16)$ (see Figure 5) in the list of the minimal forbidden subgraphs for the projective plane (see Appendix A in [15]).

Therefore, (2) is true for $k = 4$. Now suppose that (2) is true for all values less than $k \geq 5$. Let $D$ be a drawing of $C(k)$ in the projective plane and we are going to show that $v(D) \geq k - 1$.

If there exists $1 \leq i \leq 3k$ such that $v_iv_{k+i}$ crosses, then by deleting $v_iv_{k+i}$, $v_{k+i}v_{2k+i}$, $v_{2k+i}v_{i}$, we obtain a drawing of a subdivision of $C(k-1)$, denote it by $D'$. By our construction, $v(D') \leq v(D) - 1$. On the other hand, $v(D') \geq k - 2$ by induction assumption. This implies $v(D) \geq k - 1$. Therefore, we may assume that $v_iv_{k+i}$ is clean in $D$ for all $1 \leq i \leq 3k$, i.e., $C_i$ is clean for all $1 \leq i \leq k$. Then by Lemma 7, we have $v(D) \geq k - 1$.

**Proof of Theorem 1.** It follows from Lemma 3, 4, 5 and 8.

4. **Proof of Lemma 7**

This section is devoted to proving Lemma 7. Throughout this section, we assume that $C_i$ is clean for $1 \leq i \leq k$, as we have assumed in Lemma 7.
For $1 \leq i \leq k$, let

$$F_i = \{v_i v_{k+i}, v_i v_{2k+i}, v_{k+i} v_{2k+i}, v_i v_{i+1}, v_{k+i} v_{k+i+1}, v_{2k+i} v_{2k+i+1}\}.$$  

Note that the set of all $F_i$ is a partition of the edge set $E$ of $C(k)$, i.e.,

$$E = \bigcup_{i=1}^{k} F_i$$

and

$$F_i \cap F_j = \emptyset$$

for $i \neq j$.

For $1 \leq i \leq k$, define

$$f_D(F_i) = v_D(F_i) + \frac{1}{2} \sum_{j \neq i} v_D(F_i, F_j).$$

Since we have assumed that each $C_i$ is clean, there are only two possible ways of drawing $C_i$, depending on whether it is contractible or not, which are shown in Figure 6(a) and 6(b).

If $C_i$ and $C_{i+1}$ are both contractible, there are three possible ways of drawing $C_i \cup C_{i+1}$ for each $i$, which are shown in Figure 7(a), 7(b) and 7(c).

![Figure 6(a). $C_i$ is contractible.](image)

![Figure 6(b). $C_i$ is non-contractible.](image)

We have the following:

**Proposition 9.** If $C_i$ and $C_{i+1}$ are drawn as in Figure 7(a) or 7(b), then $f_D(F_i) \geq 1$.

**Proof.** Suppose $f_D(F_i) < 1$. By (4), $v_i v_{i+1}, v_{k+i} v_{k+i+1}, v_{2k+i} v_{2k+i+1}$ do not cross each other. If $C_i \cup C_{i+1}$ is drawn as in Figure 7(a), $F_i \cup C_{i+1}$ must be drawn as in Figure 8 since $C_i, C_{i+1}$ are clean and $v_i v_{i+1}, v_{k+i} v_{k+i+1}, v_{2k+i} v_{2k+i+1}$ do not cross each other. Since $C_{i-1}$ is clean, $C_{i-1}$ must lies entirely in one of the regions $f_1, f_2$ or $f_3$. We may assume that $C_{i-1}$ lies in the region $f_1$, for other cases the proof is the same. If $C_{i-1}$ lies in $f_1$, then $v_{k+i} v_{k+i+1}$ must cross $v_i v_{i+1}$ or $v_{2k+i} v_{2k+i+1}$. On the other hand, the path $v_{k+i+1} v_{k+i+2} \cdots v_{2k-i-1}$ must cross $v_i v_{i+1}$ or $v_{2k+i} v_{2k+i+1}$. Hence, by (4), $f_D(F_i) \geq 1$. Similarly, one can show that $f_D(F_i) \geq 1$ if $C_i \cup C_{i+1}$ is drawn as in Figure 7(b).
By Proposition 9, again we have Figure 7(c). In the latter case, by Proposition 10, either Corollary 11.

Combining Proposition 9 and 10, we have the following:

**Proposition 10.** If $C_i \cup C_{i+1}$ is drawn as in Figure 7(c) and $f_D(F_i) < 1$, then $F_i \cup C_{i+1}$ must be drawn as in Figure 9(b).

**Proof.** Since $f_D(F_i) < 1$, by (4), $v_{k+i}v_{k+i+1}, v_{2k+i}\cup v_{2k+i+1}$ do not cross each other. Then $F_i \cup C_{i+1}$ must be drawn as in Figure 9(a) or 9(b). If $F_i \cup C_{i+1}$ is drawn as in Figure 9(a), then $C_{i-1}$ must lie entirely in one of the regions $f_1$, $f_2$ or $f_3$ since $C_{i-1}$ is clean. We may assume that $C_{i-1}$ lies in the region $f_1$, for other cases the proof is the same. If $C_{i-1}$ lies in $f_1$, then $v_{i-1}v_i$ must cross $v_{k+i}v_{k+i+1}$ or $v_{2k+i}v_{2k+i+1}$ since $C_i$ and $C_{i+1}$ are clean. On the other hand, the path $v_{i+1}v_{i+2} \cdots v_{k-1}$ must cross $F_i$. Hence, by (4), we have $f_D(F_i) \geq 1$, which contradicts that $f_D(F_i) < 1$.

Combining Proposition 9 and 10, we have the following:

**Corollary 11.** If $F_i \cup C_{i+1}$ is not drawn as in Figure 9(b), then $f_D(F_i) \geq 1$.

**Proof.** By Proposition 10, either $f_D(F_i) \geq 1$ or $C_i \cup C_{i+1}$ is not drawn as in Figure 7(c). In the latter case, $C_i \cup C_{i+1}$ must be drawn as in Figure 7(a) or 7(b). By Proposition 9, again we have $f_D(F_i) \geq 1$. 

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**Figure 7(a)**

**Figure 7(b)**

**Figure 7(c)**

**Figure 8**

**Figure 9(a)**

**Figure 9(b)**
Remark 12. Hereafter, we say that $F_j \cup C_{j+1}$ is drawn as in Figure 9(b) if it is drawn as in Figure 9(c), i.e., replacing all the indices $i$ by $j$.

![Figure 9(c)](image)

![Figure 9(d)](image)

Figure 9(c) and Figure 9(d)

Figure 10. $F_i \cup C_{i+1} \cup F_j \cup C_{j+1}$.

Figure 11. $F_1 \cup F_2 \cup C_3$.

Proposition 13. Suppose that $F_i \cup C_{i+1}$ is drawn as in Figure 9(b). If $j \neq i-1, i, i+1$ such that $F_j \cup C_{j+1}$ is drawn as in Figure 9(b), then $F_i$ and $F_j$ must cross each other. In particular, we have $f_D(F_i) \geq 1/2$ and $f_D(F_j) \geq 1/2$.

Proof. Note that two non-contractible curves in the projective plane must cross each other. Since $F_i \cup C_{i+1}$ and $F_j \cup C_{j+1}$ are drawn as in Figure 9(b) where $j \neq i-1, i+1$, $F_i$ and $F_j$ must cross each other since $C_i, C_{i+1}, C_j, C_{j+1}$ are clean. See Figure 10 for a possible drawing of $F_i \cup C_{i+1} \cup F_j \cup C_{j+1}$. Since $F_i$ and $F_j$ cross each other, we have $v_D(F_i, F_j) \geq 1$, which implies that $f_D(F_i) \geq 1/2$ and $f_D(F_j) \geq 1/2$ by (4).
Here is the outline of the proof of Lemma 7. We will consider two cases:

Case 1. $C_i$ is contractible for all $1 \leq i \leq k$.

Case 2. $C_i$ is non-contractible for some $1 \leq i \leq k$.

For Case 1, by simple arguments, we can show that $F_1 \cup C_2$ is drawn as in Figure 9(b). Moreover, we can show that $f_D(F_{i_0}) < 1$ for some $i_0 \neq 1$. Then we will consider two cases:

Case 1.1. $i_0 \neq 2, k$.

Case 1.2. $i_0 = 2$ or $k$.

Case 1.1 can be solved easily. For Case 1.2, we will assume that $i_0 = 2$ since the proof for $i_0 = k$ is the same. Then we will consider two cases:

Case 1.2.1. $f_D(F_j) \geq 1$ for all $j \neq 1, 2$.

Case 1.2.2. $f_D(F_j) < 1$ for some $j \neq 1, 2$.

For Case 1.2.1, by assumption, $f_D(F_j) \geq 1$ for all $j \neq 1, 2$. We just need to show that $f_D(F_1) + f_D(F_2) > 0$, which implies that $v(D) = \sum_{j=1}^{k} f_D(F_j) = f_D(F_1) + f_D(F_2) + \sum_{j \neq 1, 2} f_D(F_j) > k - 2$, and hence $v(D) \geq k - 1$ since $v(D)$ is an integer. For Case 1.2.2, by assumption, $f_D(F_j) < 1$ for some $j \neq 1, 2$. Then we will consider two cases:

Case 1.2.2.1. $j \neq 3, k$.

Case 1.2.2.2. $j = 3$ or $k$.

Case 1.2.2.1 can be solved easily.

For Case 1.2.2.2, we can assume that

(5) \[ f_D(F_l) \geq 1 \text{ for } l \neq 1, 2, 3, k. \]

Otherwise, if $f_D(F_l) < 1$ for some $l \neq 1, 2, 3, k$, then it can be reduced to Case 1.2.2.1 by taking $j = l$. By simple arguments, we can reduce it to the case when both $F_3 \cup C_4$ and $F_k \cup C_1$ are drawn as in Figure 9(b). That is to say, $F_i \cup C_{i+1}$ is drawn as in Figure 9(b) for $i = 1, 2, 3, k$. Then by Proposition 13, $F_1$ crosses $F_3$ and $F_2$ crosses $F_k$. Moreover, if $k \geq 5$, then $F_1$ also crosses $F_k$. All these imply

(6) \[ f_D(F_1) \geq 1, f_D(F_k) \geq 1, f_D(F_2) \geq 1/2, \text{ and } f_D(F_3) \geq 1/2. \]

Combining (5) and (6), we get $v(D) \geq k - 1$. For $k = 4$, we will use different arguments by making use the fact that $F_i \cup C_{i+1}$ is drawn as in Figure 9(b) for $i = 1, 2, 3, 4$.

Now we are ready to prove Lemma 7.
Proof of Lemma 7. By (1), (3) and (4), the total number of crossing of the drawing \( D \) is \( v(D) = v_D(E) = \sum_{i=1}^{k} f_D(F_i) \). Therefore, it suffices to prove that \( \sum_{i=1}^{k} f_D(F_i) \geq k - 1 \). To prove by contradiction, we assume that

\[
\sum_{i=1}^{k} f_D(F_i) < k - 1.
\]

We will consider two cases: Case 1. \( C_i \) is contractible for all \( 1 \leq i \leq k \) and Case 2. \( C_i \) is non-contractible for some \( 1 \leq i \leq k \).

Case 1. Since we have assumed that \( C_i \) is clean for \( 1 \leq i \leq k \), as we have said at the beginning of this section, there are three possible ways of drawing \( C_i \cup C_{i+1} \) for each \( i \), which are shown in Figure 7(a), 7(b) or 7(c).

Note that (7) implies that \( f_D(F_i) < 1 \) for some \( i \). Without loss of generality, we may assume \( i = 1 \), i.e.,

\[
f_D(F_1) < 1.
\]

By Proposition 9, \( C_1 \cup C_2 \) must be drawn as in Figure 7(c). Hence, by (8) and Proposition 10, \( F_1 \cup C_2 \) is drawn as in Figure 9(b) (see Figure 9(d)).

There exists \( i_0 \neq 1 \) such that \( F_{i_0} \cup C_{i_0+1} \) is drawn as in Figure 9(b). (Otherwise, if \( F_j \cup C_{j+1} \) is not drawn as in Figure 9(b) for all \( j \neq 1 \), \( f_D(F_j) \geq 1 \) for all \( j \neq 1 \) by Corollary 11, which implies \( \sum_{j=1}^{k} f_D(F_j) \geq \sum_{j \neq 1} f_D(F_j) \geq k - 1 \).)

We will consider two cases: Case 1.1. \( i_0 \neq 2, k \) and Case 1.2. \( i_0 = 2 \) or \( k \).

Case 1.1. If \( i_0 \neq 2, k \), i.e., \( C_{i_0} \cup C_{i_0+1} \) is drawn as in Figure 9(b) for some \( i_0 \neq 1, 2, k \), then by Proposition 13, \( F_1 \) and \( F_{i_0} \) cross each others,

\[
f_D(F_1) \geq 1/2 \text{ and } f_D(F_{i_0}) \geq 1/2.
\]

Moreover, if there exists \( j \neq 1, 2, i_0, k \) such that \( f_D(F_j) < 1 \), then \( F_j \cup C_{j+1} \) must be drawn as in Figure 9(b) by Proposition 10. By Proposition 13, \( F_j \) and \( F_1 \) must also cross each other. Hence, \( f_D(F_1) \geq 1 \) since \( F_1 \) crosses both \( F_{i_0} \) and \( F_j \), which contradicts (8). Therefore,

\[
f_D(F_j) \geq 1 \text{ for all } j \neq 1, 2, i_0, k.
\]

Moreover, we can assume that

\[
f_D(F_2) \geq 1 \text{ and } f_D(F_k) \geq 1.
\]

(Otherwise, \( f_D(F_2) < 1 \) or \( f_D(F_k) < 1 \) implies that \( F_2 \cup C_3 \) or \( F_k \cup C_1 \) is drawn as in Figure 9(b) by Proposition 10. Replacing \( i_0 \) by 2 or \( k \), one can reduce this to Case 1.2.) Combining (9), (10) and (11), we have \( \sum_{j=1}^{k} f_D(F_j) \geq f_D(F_1) + f_D(F_{i_0}) + \sum_{j \neq 1, i_0} f_D(F_j) \geq k - 1 \).
Case 1.2. If \( i_0 = 2 \) or \( k \), then we may assume that \( i_0 = 2 \) since the proof for \( i_0 = k \) is the same. Then \( F_2 \cup C_3 \) is drawn as in Figure 9(b). We will consider two cases: Case 1.2.1. \( f_D(F_j) \geq 1 \) for all \( j \neq 1, 2 \) and Case 1.2.2. \( f_D(F_j) < 1 \) for some \( j \neq 1, 2 \).

Case 1.2.1. By assumption,

\[
(12) \quad f_D(F_j) \geq 1 \text{ for all } j \neq 1, 2.
\]

If we can show that

\[
(13) \quad f_D(F_1) + f_D(F_2) > 0,
\]

then by (12) and (13),

\[
v(D) = \sum_{j=1}^{k} f_D(F_j) = f_D(F_1) + f_D(F_2) + \sum_{j \neq 1, 2} f_D(F_j) > k - 2,
\]

which implies that \( v(D) \geq k - 1 \) since the total number of crossing \( v(D) \) is an integer.

Suppose (13) is not true, i.e.,

\[
(14) \quad f_D(F_1) = f_D(F_2) = 0.
\]

Recall that \( F_1 \cup C_2 \) is drawn as in Figure 9(d). Since \( C_3 \) is clean, \( C_3 \) must lie entirely in regions \( f_1 \) or \( f_2 \) in Figure 9(d). If \( C_3 \) lies in \( f_1 \), then \( v_2v_3 \) must cross \( v_{k+1}v_{k+2} \) or \( v_{2k+1}v_{2k+2} \). By (4), \( f_D(F_2) \geq 1/2 \), which contradicts (14). Therefore, \( C_3 \) lies in \( f_2 \). By (4) and (14), \( v_2v_3, v_{k+2}v_{k+3}, v_{2k+2}v_{2k+3} \) are clean. Then the only possible drawing of \( F_1 \cup F_2 \cup C_3 \) is shown as in Figure 11. (It is true up to renaming the vertices. For example, it is possible for \( F_1 \cup F_2 \cup C_3 \) to be drawn as in Figure 12. But one can reduce it to Figure 11 by the transformation \( v_j \mapsto v_{j-k} \).)
Since $C_4$ is clean, it must lie entirely in one of the regions in Figure 11. Note that $v_3, v_{k+3}$ and $v_{2k+3}$ do not lie in the same region in Figure 11. No matter which region $C_4$ lies in Figure 11, one of the edges $v_3v_4, v_{k+3}v_{k+4}$ and $v_{2k+3}v_{2k+4}$ must cross the $F_1$ or $F_2$ (Note that $k \geq 4$ is crucial here for $C_4$ being not equal to $C_1$). Hence, $f_D(F_1) + f_D(F_2) > 0$ which gives (13).

Case 1.2.2. If $f_D(F_j) < 1$ for some $j \neq 1, 2$, then $F_j \cup C_{j+1}$ must be drawn as in Figure 9(b) by Proposition 10. We will consider two cases: Case 1.2.2.1. $j \neq 3, k$ and Case 1.2.2.2. $j = 3$ or $k$.

Case 1.2.2.1. Since $F_j \cup C_{j+1}$ is drawn as in Figure 9(b) where $j \neq 1, 2, 3, k, F_j$ must cross $F_1$ and $F_2$ by Proposition 13, since $F_1 \cup C_2$ and $F_2 \cup C_3$ are drawn as in Figure 9(b). This implies that, by (4),

\[(15) \quad f_D(F_1) \geq 1/2, f_D(F_2) \geq 1/2, \text{ and } f_D(F_j) \geq 1.\]

Note that

\[(16) \quad f_D(F_r) \geq 1 \text{ for all } r \neq 1, 2, 3, j, k.\]

Otherwise, if $f_D(F_r) < 1$ for some $r \neq 1, 2, 3, j, k$, then by Proposition 10, $F_r \cup C_{r+1}$ is drawn as in Figure 9(b). By Proposition 13, $F_r$ also crosses $F_1$. This implies $f_D(F_1) \geq 1$ since $F_1$ cross $F_j$ and $F_r$, which contradicts (8).

We claim that

\[(17) \quad f_D(F_3) \geq 1 \text{ and } f_D(F_k) \geq 1.\]

To see this, suppose that $f_D(F_3) < 1$. Then $F_3 \cup C_4$ is drawn as in Figure 9(b) by Proposition 10. Hence $F_1$ must cross $F_3$ and $F_j$ by Proposition 13, which implies that $f_D(F_1) \geq 1$ and contradicts (8). On the other hand, if $f_D(F_k) < 1$, then $F_k \cup C_1$ must be drawn as in Figure 9(b) by Proposition 10. Hence $F_2$ must cross $F_k$ and $F_j$ by Proposition 13, which implies that $f_D(F_2) \geq 1$ and contradicts (8). This proves (17).

Combining (15), (16) and (17), we get $\sum_{r=1}^{k} f_D(F_r) = f_D(F_1) + f_D(F_2) + \sum_{r \neq 1, 2} f_D(F_r) \geq k - 1$.

Case 1.2.2.2. If $j = 3$ or $k$, then $F_k \cup C_1$ or $F_3 \cup C_4$ is drawn as in Figure 9(b). We may assume that

\[(18) \quad f_D(F_l) \geq 1 \text{ for } l \neq 1, 2, 3, k.\]

(Otherwise, if $f_D(F_l) < 1$ for some $l \neq 1, 2, 3, k$, then it can be reduces to Case 1.2.2.1 by taking $j = l$.) It can be reduced to the case when both $F_3 \cup C_4$ and $F_k \cup C_1$ are drawn as in Figure 9(b).
To see this, suppose that $F_3 \cup C_4$ is drawn as in Figure 9(b) and $F_k \cup C_1$ is not. Then by Corollary 11

(19) \[ f_D(F_k) \geq 1, \]

and $F_3$ must cross $F_1$ by Proposition 13 since $F_1 \cup C_2$ is drawn as in Figure 9(b). We claim that $F_1$ must cross $F_k$. Assuming the claim, we have

(20) \[ f_D(F_1) \geq 1 \text{ and } f_D(F_3) \geq 1/2. \]

Combining (18), (19) and (20), we get $\sum_{r=1}^{k} f_D(F_r) > k - 2$, which implies that $v(D) = \sum_{r=1}^{k} f_D(F_r) \geq k - 1$ since $v(D)$ is an integer.
To show the claim, i.e., $F_1$ crosses $F_k$, we note that $F_1 \cup C_2$ is drawn as in Figure 9(b). See Figure 13. Since $C_k$ is clean, it must lie entirely in one of the regions in Figure 13. It is impossible for $C_k$ to lie in $f_3$, otherwise, the path $v_2v_3\cdots v_k$ crosses $C_1$. It is also impossible for $C_{-1}$ to lie in $f_4$, otherwise, $v_kv_{k+1}$ crosses $C_2$. If $C_k$ lies in $f_1$, $v_3v_1$ must cross with $v_{k+1}v_{k+2}$ or $v_{2k+1}v_{2k+2}$, which implies that $F_k$ crosses $F_1$. If $C_k$ lies in $f_2$, then $F_k$ must cross $F_1$ since $F_k \cup C_1$ is not drawn as in Figure 9(b) by our assumption (See Figure 14 for example). Therefore, $F_1$ must cross $F_k$, as we claimed.

Similarly, if $F_k \cup C_1$ is drawn as in Figure 9(b) and $F_3 \cup C_4$ is not, then $\sum_{r=1}^{k} f_D(F_r) \geq k - 1$.

We will show that $v(D) \geq 3$. By contradiction, suppose that $v(D) \leq 2$. By (1) and (22), we have

$$f_D(F_1) = f_D(F_2) = f_D(F_3) = f_D(F_4) = 1/2.$$  

Since $F_1$ crosses $F_3$, by (4) and (23) we get

$$v_D(F_1, F_3) = 1, v_D(F_1, F_j) = 0 \text{ for } j \neq 3, v_D(F_3, F_j) = 0 \text{ for } j \neq 1.$$  

Similarly, since $F_2$ crosses $F_4$, by (4) and (23) we get

$$v_D(F_2, F_4) = 1, v_D(F_2, F_j) = 0 \text{ for } j \neq 4, v_D(F_4, F_j) = 0 \text{ for } j \neq 2.$$  

Since $F_1 \cup C_2$ and $F_3 \cup C_4$ are drawn as in Figure 9(b), the only possible drawing of $F_1 \cup C_2 \cup F_3 \cup C_4$ is shown in Figure 15(a) in view of (24) and (25). However, one can show that it is impossible for (24), (25) to hold. For example, if $F_1 \cup C_2 \cup F_3 \cup C_4$ is drawn in Figure 15(b), then the edge $v_8v_9$ must cross with $F_1$ or $F_3$, which contradicts (24); and if $F_1 \cup C_2 \cup F_3 \cup C_4$ is drawn in Figure 15(c), then the edge $v_2v_3$ must lie entirely in the region $f$, as in Figure 15(d), since $v_D(F_2, F_j) = 0 \text{ for } j \neq 4$ by (25). However, in Figure 15(d), no matter how $v_6v_7$
is drawn, \( v_6v_7 \) must either (i) cross \( v_2v_3 \) which contradicts (25), or (ii) cross \( C_i \) which contradicts that \( C_i \) are all clean, or (iii) cross \( F_1 \) or \( F_3 \) which contradicts (25). We leave other cases to the reader.

**Case 2.** If there exists \( 1 \leq i \leq k \) such that \( C_i \) is non-contractible, then we may assume that \( C_1 \) is non-contractible. Then \( C_i \) is contractible for all \( i \neq 1 \). (Otherwise, \( C_i \) crosses \( C_1 \) since two non-contractible curves in the projective plane must cross each other. This contradicts the assumption that all \( C_i \) are clean.) Since \( C_i \) and \( C_{i+1} \) are clean and contractible for \( i \neq 1, k \), there are three possible ways of drawing \( C_i \cup C_{i+1} \), which are shown in Figure 7(a), 7(b) or 7(c).

We claim that

\[
(26) \quad f_D(F_i) \geq 1 \text{ for } i \neq 1, k.
\]

To prove this, suppose that \( f_D(F_i) < 1 \) for some \( i \neq 1, k \). By Corollary 11, \( F_i \cup C_{i+1} \) must be drawn as in Figure 9(b), which crosses the non-contractible \( C_1 \). This contradicts that \( C_1 \) is clean. This proves (26).

Now we are going to show that

\[
(27) \quad f_D(F_1) + f_D(F_k) > 0.
\]

Combining this with (26), we will get \( \sum_{r=1}^{k} f_D(F_r) > k - 2 \), which gives \( v(D) = \sum_{i=1}^{k} f_D(F_i) \geq k - 1 \) since \( v(D) \) is an integer. Suppose that (27) is not true, i.e.,

\[
(28) \quad f_D(F_1) = f_D(F_k) = 0.
\]

Since \( C_1 \) is non-contractile and \( C_2 \) is contractible, \( C_1 \cup C_2 \) must be drawn as in Figure 16. On the other hand, by the same reasons, \( C_1 \cup C_k \) must be drawn as in Figure 16 by replacing \( C_2 \) by \( C_k \).

By (4) and (28), \( v_1v_2, v_{k+1}v_{k+2}, v_{2k+1}v_{2k+2} \) do not cross. From Figure 16, one can see that there are three possible ways of drawing \( F_1 \cup C_2 \), which are shown in Figure 17(a), 17(b) and 17(c).
If $F_1 \cup C_2$ is drawn as in Figure 17(b) and 17(c), then $C_3$ must lie entirely in one of the regions since $C_3$ is clean. Then $F_2$ must cross with $F_1$ since there is no region in Figure 17(b) or 17(c) containing all of the vertices $v_2$, $v_{k+2}$ and $v_{2k+2}$. This implies $f_D(F_1) > 0$, which contradicts (28).

Therefore, $F_1 \cup C_2$ must be drawn as in Figure 17(a). By the same argument, $F_k \cup C_1$ must be drawn as in Figure 17(a) by replacing $C_2$ by $C_k$. Hence, $F_k \cup F_1 \cup C_2$ must be drawn as in Figure 18(a) or 18(b) since $F_1$ does not cross $F_k$ by (28).

Note that $C_3$ must lie in one of the regions in Figure 18(a) or 18(b). Since there exists no region in Figure 18(a) or 18(b) which contains all of the vertices $v_2$, $v_{k+2}$ and $v_{2k+2}$, $F_3$ must cross either $F_k$ or $F_1$ ($k \geq 4$ is needed here for $F_3$ being not equal to $F_k$). This implies that $f_D(F_1) > 0$ or $f_D(F_k) > 0$, which gives (27).

This finishes the the proof of Lemma 7.
References


Received 2 September 2010
Revised 26 January 2011
Accepted 26 January 2011